Compactness and Symmetry in Quantum Logic

Alexander Wilce
Department of Mathematics, Susquehanna University,
Selinsgrove, PA 17870, USA
E-mail: wilce@susqu.edu

A particularly simple and flexible mathematical framework for the study of probability theory - classical, quantum and otherwise - is the notion of a test space, i.e., a collection of (possibly overlapping) discrete sample spaces (as developed in the 1970s and 80s by D. J. Foulis, C. H. Randall and others). "Quantum logics" arise quite naturally as invariants of test spaces; however, the latter are much easier both to interpret and to manipulate. After providing a tutorial on test spaces, I'll outline how this framework can usefully be enriched by the addition of topological and covariant structure. In particular, I'll discuss topological test spaces that are highly symmetrical under the action of a compact group

1 Introduction

Minimally interpreted, quantum mechanics is a non-classical probability calculus. Indeed, one can essentially recover the entire apparatus of quantum mechanics from the single postulate that the logic of probability-bearing "events" has the structure of the projection lattice of a Hilbert space (or, more generally, of a von Neumann algebra), with commuting projections corresponding to *jointly observable* events, and orthogonal projections, to mutually exclusive events.[a] However, from the perspective of probability theory, once we allow that not all events need to be jointly observable, a host of formal possibilities present themselves for the structure of the set of events – most of these far removed from projection lattices.

Thus we come up against the question, what's so special about the particular mathematical framework of quantum probability theory? One possible answer, of course, is naturalistic, i.e., *that's just how the world is.* But many people have harbored the suspicion that more could be said: that there may be something quite *canonical* about the formal apparatus of quantum probability theory. This idea is already palpable in von Neumann's work, and has motivated a great deal of research since. In particular, a large part of the literature on quantum logic amounts to a sustained attack on this problem.[b]

[a]This idea was put forward in von Neumann's monograph of 1932 [19], and later reinforced by the work of Mackey [14], Gleason [6] and others in the 1950s. For details, see the book of Varadarajan [18].
[b]A number of speakers at this conference – L. Hardy [9], I. Helland [11], A. Khrennikov [13],

If one is serious about asking whether and in what sense quantum probability is canonical, or natural, one should first sharpen the question by asking: *compared to what*? In other words, we need a general mathematical framework for discussing probabilistic models – classical, quantum, or otherwise. A particularly elementary, but still very general and flexible, framework of this sort, developed mainly by D. J. Foulis and C. H. Randall in the 1970s and 80s, is based on the notion of a *test space*: a collection of (possibly overlapping) discrete classical sample spaces. "Quantum logics" in the technical sense, e.g., orthomodular lattices and posets, arise naturally as invariants of test spaces; however, the latter are much easier both to interpret and to manipulate.

The first part of this paper is wholly tutorial. My goal here is not so much to present a systematic overview of the formalism of test spaces, as to give the reader a taste of – and, I hope, a taste *for* – this approach to generalized probability theory. More detailed expositions can be found in [3 4 5 8 22]. One of the most striking differences between classical and quantum probability theory is that the mathematical setting for the latter – the unit sphere of a Hilbert space – has an altogether richer topological and group-theoretic structure than that of the former. In the second part of the paper, I indicate how the basic apparatus of test spaces can be enriched by the addition of topological and group-theoretic structure. In particular, I'll discuss the structure of test spaces that are highly symmetric under the action of a compact group.

2 Test Spaces

A *test space* is a pair $(X, \mathfrak{A})$ where X is a non-empty set and $\mathfrak{A}$ is a covering of X by non-empty subsets, called *tests*. The intended interpretation is that each set $E \in \mathfrak{A}$ represents the set of all possible *outcomes* of some experiment, decision, or physical process. Thus, a test space is a generalization of the notion of a discrete sample space in classical probability theory. Accordingly, we call a subset of a test an *event*. A *probability weight* or *state* on $(X, \mathfrak{A})$ is a function $\omega : X \to [0, 1]$ summing to 1 over each test. A (simple) *random variable* on $(X, \mathfrak{A})$ is a mapping $f : E \to \mathbb{R}$ defined on a test $E \in \mathfrak{A}$.

Note that we permit distinct tests to overlap. For later reference, a test space in which distinct tests are disjoint is said to be *semiclassical*. However, as the examples below illustrate, non-semiclassical test spaces arise quite naturally. Notice also that the probability of an outcome with respect to a given state is test-independent. In other words, the probabilities that states assign to outcomes are *non-contextual*.

to mention a few – have discussed work directed toward much the same goal, albeit along different lines.

2.1 A Two-Bit Example: Consider two (non-identical) coins, one with faces H and T, the other with faces h and t. Let E be the experiment of flipping the first coin, so that $E = \{H, T\}$; let F be the experiment of flipping the second coin, so that $F = \{h, t\}$. We have then a very simple semi-classical test space with $X = \{H, T, h, t\}$ and $\mathfrak{A} = \{E, F\} = \{\{H, T\}, \{h, t\}\}$. We can now consider *sequential* experiments such as the following: (a) Flip coin E twice in succession, or (b) flip coin E: if the result is H, flip it again; if not, flip coin F. The outcome-sets here are $\{HH, HT, TH, TT\}$ and $\{HH, HT, Th, Tt\}$. Note that these have two outcomes in common, namely HH and HT.

2.2 Classical test spaces. Discrete classical probability theory is the theory of test spaces of the form $(E, \{E\})$, having just a single test. A bit more generally, let S be a set and Σ, a field of subsets of S. Let $\mathcal{B} = \mathcal{B}(S, \Sigma)$ be the collection of (say, countable) partitions of S into non-empty Σ-measurable sets. We can regard each partition $E \in \mathcal{B}$ as the outcome-set for a "coarse-grained" measurement of a value in S.

Accordingly, we have a test space $(\Sigma^*, \mathcal{B}(S, \Sigma))$, where Σ^* is the set of non-empty elements of Σ. This is called the *Borel test space* associated with (S, Σ). States on $(\Sigma^*, \mathcal{B}(S, \Sigma))$ correspond in an obvious way to σ-additive probability measures on (S, Σ), while simple random variables on $(\Sigma^*, \mathcal{B}(S, \Sigma))$ correspond to simple random variables (in the usual sense) on (S, Σ); more general random variables can be recovered as limits of these. Thus, Kolmogorovian probability theory is essentially the theory of Borel test spaces.

2.3 Quantum test spaces Let $\mathbf{H}$ be a Hilbert space. Let $S = S_{\mathbf{H}}$ be $\mathbf{H}$'s unit sphere and $\mathfrak{F} = \mathfrak{F}_{\mathbf{H}}$, the set of *frames* of $\mathbf{H}$, i.e., the set of maximal orthonormal subsets of $\mathbf{H}$. Then $(S, \mathfrak{F})$ is a test space, representing the collection of (maximal) discrete quantum-mechanical experiments. Gleason's theorem [6] lets us represent probability weights by density operators in the usual way, i.e., $\omega(x) = \langle Wx, x \rangle$. The spectral theorem lets us represent bounded random variables on $(S, \mathfrak{F})$ by bounded self-adjoint operators with discrete spectra (from which more general observables can be recovered as suitable limits). Thus, quantum probability theory is essentially the theory of quantum test spaces.

One of the virtues of test spaces is the ease with which one can manufacture "toy models", i.e., simple and instructive *ad hoc* examples. The following is a well-known example (due originally to Ron Wright) which I rehearse here in the expectation that it will be unfamiliar to many readers.

2.4 Example: The "Wright Triangle". Consider a sealed triangular box with opaque top and bottom and translucent walls. The interior is divided into three chambers, each chamber occupying one corner, as in figure 1 below. Inside the box is a firefly, which is visible when viewed through a given wall if, and only if, the firefly occupies one of the two chambers behind that wall, and is flashing.

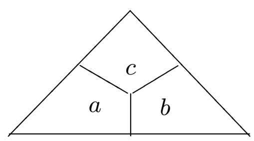

Figure 1

Each wall corresponds to an experiment. Looking through the south-facing wall, we may see a light in chamber a, a light in chamber b, or we may see no light at all. Representing this latter outcome by x, we may represent the experiment of looking through the south wall by $\{a, x, b\}$. Representing the experiments associated with the other two walls similarly by $\{b, y, c\}$ and $\{c, z, a\}$, we have a test space $\{\{a, x, b\}, \{b, y, c\}, \{c, z, a\}\}$. This can conveniently be represented by a graph, as in figure 2. Here each node represents an outcome, as indicated, with the outcomes corresponding to each experiment lying along a common line.

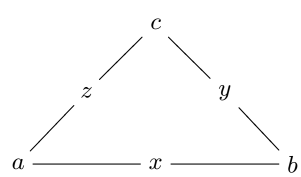

Figure 2

2.5 Greechie Diagrams The graphical convention we used in discussing the Wright Triangle is an example of a useful device, due to R. Greechie, for representing small finite test spaces. The idea is to represent each outcome by a node in a graph, connecting the outcomes belonging to a given test along a smooth arc (e.g., a straight line or, if necessary, some other smooth curve), so arranging matters that the arcs corresponding to distinct but overlapping tests intersect one another transversally, so that they can readily be distinguished from one another by eye. Here are some simple examples in which the outcomes comprising a test lie along a straight line.

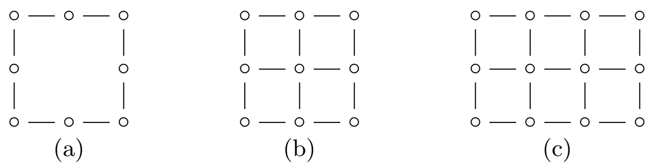

Figure 3

Figure 3 (a) illustrates four three-outcome tests pasted together in a loop. (Figure 3 (b) shows a test space sometimes called the *3-by-3 window*, the states on which correspond in an obvious way to the three-by-three doubly-stochastic matrices. Figure 3 (c) shows a state space that admits *no* states.

A model for the 4-loop of figure 3 (a) would be a "firefly-box" having four, rather than three chambers. Here is a concrete model for the three-by-three window of Figure 3(b) [21]. Let $E = \{x, y, z\}$ and $F = \{a, b, c\}$ be two disjoint 3-outcome tests. Let $f : E \to F$ be any bijection – say, $f = \{(x, a), (y, c), (z, b)\}$. Consider the following experiment: choose one of the tests E and F, perform it, and record as your outcome the unique pair in the graph of f having the observed outcome as a member. For instance, performing E and observing outcome y, record as the outcome of the experiment the pair (y, c). Evidently, the outcome-set for this experiment is just the graph of f. Each of the six bijection $E \to F$ yields a similar experiment. The resulting test space is displayed, Greechie-style, in figure 4 below. Greechie-style:

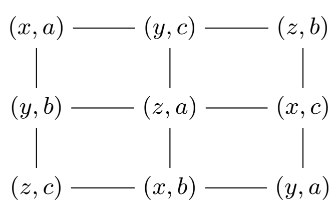

Figure 4

2.6 Dispersion-Free States Call a state ω on a test space *dispersion free* iff it takes only the values 0 and 1. Equivalently, we may think of a dispersion-free state as a *transversal* of the set of tests, i.e., a subset of X meeting each test exactly once. For example, the dispersion-free states on the Wright triangle are the four transversals pictured below. The first three of these correspond in an obvious way to the situations in which the firefly is flashing in one of the three chambers; the third describes the situation in which the firefly is not flashing at all (in which case its location is unknown).

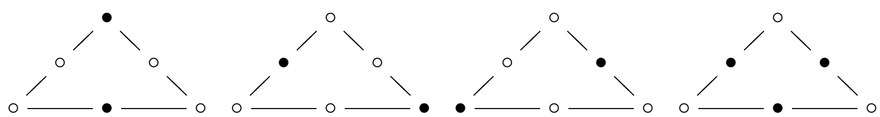

Figure 5

Except for the quantum test spaces $(X_{\mathbf{H}}, \mathfrak{F}_{\mathbf{H}})$, which has none at all, each of the test spaces considered above support a lavish supply of dispersion-free states. Indeed, in each case one can find for each outcome $x \in X$ a dispersion-free state ω with $\omega(x) = 1$. A test space with this feature is said to have a *unital* set of dispersion-free states, or to be *UDF*, for short. It is not hard to see that a test space is UDF if and only if it is equivalent to a space of partitions, i.e., a sub-test space of a Borel test space (as defined in Example 2.2).

As just illustrated, the Wright triangle is UDF; so is the three-by-three window, the dispersion-free states on which correspond to the permutation matrices. As is well-known, the latter are the extreme points of the doubly-stochastic 3×3 matrices, so in this instance every state is a weighted average, or mixture, of dispersion-free states. This is not generally the case. Consider the state $\omega(a) = \omega(b) = \omega(c) = 1/2$, $\omega(x) = \omega(y) = \omega(z) = 0$ on the Wright triangle. This is an extreme state, but certainly not an average of the four dispersion-free states discussed above.[c]

3 Quantum Logics

Traditionally, quantum logic has been associated with the study of various order-theoretic and partial-algebraic structures abstracted from the the lattice of projections on a Hilbert space. These include, in order of increasing generality, orthomodular lattices and posets, orthoalgebras, and effect algebras. From the point of view advanced here, these should not be taken as fundamental structures, but rather, as natural and useful *invariants* of certain classes of test spaces.

It may be in order to recall some order-theoretic ideas [12]. Let $(L, \leq, 0, 1)$ be a poset with least element 0 and greatest element 1. An *orthocomplementation* on L is a mapping $' : L \to L$ satisfying, for all $a, b \in L$, (i) $a \leq b \Rightarrow b' \leq a'$, (ii) $a \leq a''$, and (iii) $a \wedge a' = 0$. If $a, b \in L$ with $a \leq b'$, we call a and b *orthogonal*,

[c]This state seems to describe a "gregarious" firefly that presents itself at whichever window it is through which the observer is peering, choosing the left or right-hand chamber at random.

and write $a \perp b$. If $a \perp b$ and the join $a \vee b$ exists, then we denote it by $a \oplus b$. L is *orthomodular* iff (i) $a \perp b \Rightarrow a \oplus b$ exists, and (ii) $a \oplus b = a \oplus c \Rightarrow b = c$ for all $a, b, c \in L$. An *orthomodular lattice* (OML) is a lattice-ordered OMP. In this context, the existence of $a \vee b$ is guaranteed, so orthomodularity comes down to the condition that whenever $a \perp b$, $(a \wedge b') \vee b = a$.

The primordial example of a non-Boolean orthomodular lattice is the lattice of closed subspaces of a Hilbert space. The celebrated Piron representation theorem [17] gives necessary and sufficient conditions for an orthomodular lattice L to be isomorphic to the lattice of ortho-closed subspaces[d] of an inner product space over a $*$-division ring. Among other things, L must be complete and atomic, and must satisfy the so-called *atomic covering law*: if x is an atom and $x \not\leq a \in L$, then $a \vee x$ must cover a, i.e., there must be no element of L properly between a and $a \vee x$. It remains unclear to what extent these conditions can be given a direct and compelling "physical" justification. Moreover, in order to recover quantum mechanics proper, one must narrow the choice of the division ring to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the Quaternions). This requires further assumptions, the physical meaning of which is also at present not perfectly clear [10].

3.1 Definition and Notation: If $(X, \mathfrak{A})$ is a test space, let $\mathcal{E} = \mathcal{E}(X, \mathfrak{A})$ denote the associated set of events. We say that two events A and B are *compatible* iff they belong to a common test (i.e., $A \cup B \subseteq E$ for some $E \in \mathfrak{A}$). We also say that A and B are

(a) *orthogonal*, writing $A \perp B$, if they are compatible and disjoint, i.e., $A \cap B = \emptyset$ and $A \cup B$ is an event;

(b) *complementary*, writing $A\mathsf{oc}B$, if they partition a test;

(c) *perspective*, writing $A \sim B$, if they share a common complementary event.

We say that $(X, \mathfrak{A})$ is *algebraic* iff for all events $A, B, C \in \mathcal{E}(X, \mathfrak{A})$,

$$A \sim B \text{ and } B\mathsf{oc}C \;\Rightarrow\; A\mathsf{oc}C.$$

3.2 Example: Consider again the Wright Triangle of Example 2.4

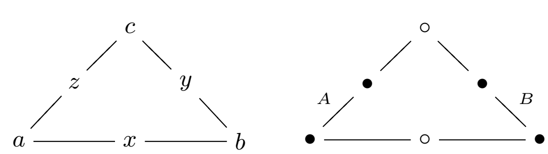

Figure 6

[d] That is, subspaces M satisfying $M = M^{\perp\perp}$

The events $A = \{a, z\}$ and $B = \{b, y\}$, represented by the shaded nodes in the second Greechie diagram above, are both complementary to the event $C = \{c\}$, hence, perspective to one another. Note that we can regard the events A and B as "physically equivalent" in so far as they both convey the information: "the firefly is not currently flashing in Room C".

All of the finite (toy) examples considered thus far are algebraic. So, too, are our benchmark examples, the Borel and quantum test spaces of Examples 2.3 and 2.4. In the former, events are countable pairwise disjoint families of measurable sets; two events are perspective precisely when they have the same union, and complementary when their unions are disjoint. In a quantum test space, events are orthonormal sets of vectors in a Hilbert space $\mathbf{H}$. Here two events are perspective when they have the same closed span, and complementary when their closed spans are complementary subspaces of $\mathbf{H}$.

3.2 Lemma: *If $(X, \mathfrak{A})$ is algebraic, then*

(a) $\sim$ is an equivalence relation on $\mathcal{E}$;

(b) If $A \perp B$ and $B \sim C$, then $A \perp C$ as well, and $A \cup B \sim A \cup C$.

If $(X, \mathfrak{A})$ is algebraic, write $p(A)$ for the $\sim$-equivalence class of an event $A \in \mathcal{E}(X, \mathfrak{A})$. The *logic* of $(X, \mathfrak{A})$ is the set

$$\Pi(X, \mathfrak{A}) = \{p(A) | A \in \mathcal{E}\}$$

of all such equivalence classes, equipped with the partially-defined binary operation

$$p(A) \oplus p(B) := p(A \cup B),$$

(well-) defined for orthogonal pairs of events. We may also define $0 := p(\emptyset)$, $1 := p(E)$, $E \in \mathfrak{A}$, and $p(A)' = p(C)$ where $C\mathsf{oc}A$. The structure $(\Pi, \oplus, ', 0, 1)$ satisfies the conditions of the following

3.3 Definition: An *orthoalgebra* is a structure $(L, \oplus, 0, 1)$ where $\oplus$ is a partial binary operation on L satisfying

(1) $p \oplus q = q \oplus p$ and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$

(2) $p \oplus p$ is defined only if $p = 0$;

(3) $p \oplus 0 = 0 \oplus p = p$;

(4) $\forall p \in L$, $\exists! p' \in L$ with $p \oplus p' = 1$.

Thus, the logic of an algebraic test space is an orthoalgebra. Conversely, any orthoalgebra L can be represented as such a logic, as follows. Let $X_L = L \setminus \{0\}$ and let $\mathfrak{A}_L$ denote the set of finite subsets $E = \{e_1, ..., e_n\}$ of $L \setminus 0$ for which $e_1 \oplus \cdots \oplus e_n$ exists and equals 1. Then $(X_L, \mathfrak{A}_L)$ is algebraic, with logic canonically isomorphic to L.

Any orthomodular poset can be regarded as an orthoalgebra. Indeed, if $(L, \leq, ')$ is any orthoposet, the partial binary operation of orthogonal join – that is, $p \oplus q = p \vee q$ for $p \leq q'$ – is associative. It is cancellative if and only if L is orthomodular, in which case, $(L, \oplus)$ is an orthoalgebra, the natural order on which agrees with the given order on L.

Conversely, any orthoalgebra carries a natural partial order, defined by setting $p \leq q$ iff there exists some $r \in L$ with $p \perp r$ and $p \oplus r = q$. With respect to this ordering, the mapping $p \mapsto p'$ is an orthocomplementation. However,$(L, \leq, ')$ will generally *not* be an OMP, owing to the fact that $p \oplus q$ is generally not the supremum of p and q in L. For instance, in Example 3.2 above, the propositions $p = p(\{a\})$ and $q = p(\{b\})$ have two distinct minimal upper bounds, namely, $p \oplus q = p(\{a, b\})$ and $p(A) = p(B)$.

3.4 Proposition [3]**:** *If L is an orthoalgebra, the following are equivalent:*

(a) *L is* orthocoherent*, i.e., for all pairwise orthogonal elements $p, q, r \in L$, $p \oplus q \oplus r$ exists.*

(b) *$p \oplus q = p \vee q$ for all $p \perp q$ in L*

(c) *$(L, \leq, ')$ is an orthomodular poset.*

Thus, orthomodular posets and orthomodular lattices can be regarded as essentially the same things as orthocoherent and lattice-ordered orthoalgebras, respectively.

3.5 Examples: The logic of the Wright triangle gives an example (the simplest one) of a non-orthocoherent orthoalgebra. For an example of an algebraic test space the logic of which is orthocoherent but not lattice ordered, consider the 4-loop (Example 2.5 (b)). Let a and b denote the outcomes in the lower left and upper right corners, as in Figure 7 (a). One can extend a and b to perspective events A and B, respectively, so that $p(a), p(b) \leq q := p(A) = p(B)$, as illustrated in figure 7 (b). A different such extension, to perspective events C and D, is indicated in figure 7 (c); again we have $p(a), p(b) \leq r := p(C) = p(D)$. It is clear that q and r are not comparable, i.e., $q \not\leq r$ and $r \not\leq q$; thus q and r are distinct minimal upper bounds for $p(a)$ and $p(b)$.

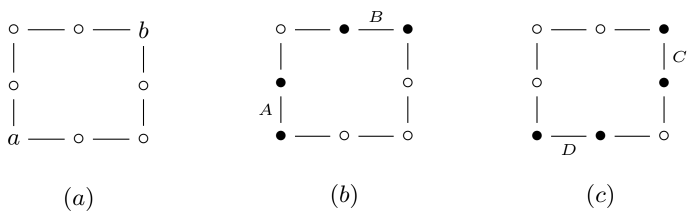

Figure 7

The following result is a special case of what is usually called the *Loop Lemma* (See Theorem 6.9 in [4] for a more general version). A finite-rank test space $(X, \mathfrak{A})$ is *coherent* iff every pairwise-orthogonal set of outcomes is an event.

3.6 Proposition: *Let* $(X, \mathfrak{A})$ *be a coherent algebraic test space of finite rank. If* $\mathfrak{A}$ *contains no* 4*-loop, then* $\Pi(X, \mathfrak{A})$ *is an orthomodular lattice.*

4 Topological Test Spaces

Both the quantum test space $(X, \mathfrak{F})$ and its associated logic $L(\mathbf{H})$ are *topological*, as well as combinatorial and order-theoretic, objects. Indeed, X and $\mathfrak{F}$ are both complex manifolds, and $L(\mathbf{H})$ is a topological sum of manifolds of different dimensions (namely, the Grassmannian manifolds of dimensions $n = 1, ..., \dim(\mathbf{H})$). In view of this, it is remarkable how little has been done in the direction of topologizing the general apparatus of quantum logic. This probably owes to the fact that the lattice-theoretic properties of $L(\mathbf{H})$ uniquely determine the Hilbert space $\mathbf{H}$ (including its scalar field), and thus also determines the topological and geometric structure just alluded to. However, one should not rule out the possibility that *assumptions* involving continuity and symmetry will be helpful in the project of characterizing standard quantum mechanics among the welter of possible statistical models.

This seems to me sufficient motivation to undertake a general study of "topological quantum logic". The proof of such a pudding is very much in the eating, of course. As it turns out, adding even a little topological seasoning to the standard quantum-logical constructions described above produces tasty results – some of which we'll now sample.

4.1 Definition: By a *topological test space* I mean a test space $(X, \mathfrak{A})$ where

(i) X is a Hausdorff space

(ii) The orthogonality relation $\perp$ is closed in $X \times X$.

Natural examples abound: the quantum test space $(X, \mathfrak{F})$ is a topological test space if we give the unit sphere X its norm topology; in a different direction, any cartesian product of discrete test spaces can be regarded as a zero-dimensional topological test space.

The requirement that the orthogonality relation be closed has the following strong consequence. Let us call a set of outcomes of which no two are orthogonal *totally non-orthogonal*, abbreviating this *TNO*.

4.2 Lemma: *Let $(X, \mathfrak{A})$ be any topological test space. Then the topology on X has a basis consisting of TNO open sets.*

Proof: Since any subset of a TNO set is again TNO, it suffices to show that every outcome $x \in X$ has a TNO open neighborhood. Since $\perp$ is closed and $(x, x) \not\in \perp$, there exists a pair of open sets U, V with $(x, x) \in U \times V$ and $U \times V \cap \perp = \emptyset$. Then $W = U \cap V$ provides the desired neighborhood. $\Box$

One immediate consequence of Lemma 4.2 is that every pairwise-orthogonal subset of X – in particular, every test – is discrete. One can also show that every test is closed [23]. Another consequence is the following easy, but perhaps somewhat surprising, result:

4.3 Proposition *Let $(X, \mathfrak{A})$ be a topological test space with X compact. Then there exists a natural number n such that any pairwise orthogonal subset of X has n or fewer members. In particular, $\mathfrak{A}$ has finite rank.*

Proof: For each $x \in X$, let U_x be a totally non-orthogonal neighborhood of x. As X is compact, finitely many of these, say $U_1, ..., U_n$, cover X. A pairwise orthogonal subset of X must select at most one element from each of these, hence, must have n or fewer elements. $\Box$.

4.1 Topologizing the space of tests

We can place a topology on the space $\mathfrak{A}$ of tests, as follows. If U is an open subset of X, let $[U]$ denote the set of tests $E \in \mathfrak{A}$ that meet U – that is,

$E \cap U \neq \emptyset$. Let (U) be the set of tests contained in U. By the *standard topology* on $\mathfrak{A}$, we mean the coarsest topology making both $[U]$ and (U) open for all open subsets of X. Topologists will recognize this as the relative Vietoris topology that $\mathfrak{A}$ inherits from the hyperspace 2^X of all closed subsets of X [16].

If $(X, \mathfrak{A})$ is locally finite, we can describe the standard topology on $\mathfrak{A}$ in fairly vivid geometrical terms. Consider a test $E \in \mathfrak{A}$. For each outcome $x \in E$, choose a totally non-orthogonal neighborhood U_x, arranging for these to be pairwise disjoint (as we can, since X is Hausdorff). Note that a test meets U_x, if at all, in just a single outcome. Thus, the open set

$$\langle U_x | x \in E \rangle \ := \ \bigcap_{x \in E} [U_x] \cap \left(\bigcup_{x \in E} U_x \right).$$

contains precisely of those tests that consist of a *selection* of exactly one outcome from each of the open sets U_x. This is illustrated in Figure 8 below.

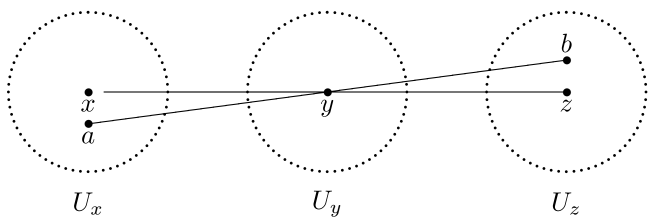

Figure 8
$E = \{x, y, z\}$; $F = \{a, y, b\} \in \langle U_x, U_y, U_z \rangle$.

Call a set of tests of this form $\langle U_x | x \in E \rangle$ a *standard neighborhood* of E. These form a basis for a topology on $\mathfrak{A}$, which I'll call the *standard topology*.[e]

Let $(X, \mathfrak{A})$ be a locally finite topological test space, and let $\mathcal{U} = \langle U_x | x \in E \rangle$ be a standard neighborhood of $E \in \mathfrak{A}$. Let $U = \bigcup \mathcal{U}$. Then

- For each test $F \in \mathcal{U}$, we have a canonical bijection $\phi : F \to E$ taking each $x \in F$ to the unique point $y = \phi(x) \in E \cap U_x$. In particular, every test $F \in \mathcal{U}$ has the same cardinality as E.

- Each U_x selects a single outcome from each test $F \in \mathcal{U}$, and hence, defines a dispersion-free state on $(U, \mathcal{U})$.

Thus, every locally finite topological test space is *locally uniform* and *locally UDF*. The following result [23], generalizing the Meyer and Clifton-Kent density

[e] Those schooled in such matters will recognize this as the relative Vietoris topology $\mathfrak{A}$ inherits from the hyperspace 2^X of closed subsets of X.

theorems [2] [15] gives a sense in which many topological test spaces are "almost" UDF, not just locally but *globally*.

4.4 Proposition *Let* $(X, \mathfrak{A})$ *be a topological test space with* X *a second-countable space having no isolated points. Then there exists a countable dense semi-classical subset of* $\mathfrak{A}$*, the union of which is dense in* X*.*

4.5 Ordered Tests If $(X, \mathfrak{A})$ is a *uniform* test space of finite rank n, there is an alternative characterization of the standard topology on $\mathfrak{A}$ that is in many respects easier to work with. Let $\mathfrak{A}^o \subseteq X^n$ be the space of all *ordered tests*, i.e., n-tuples $(x_1, ..., x_n)$ with $\{x_1, ..., x_n\} =: E \in \mathfrak{A}$, and let $\pi : \mathfrak{A}^o \to \mathfrak{A}$ be the surjection $(x_1, ..., x_n) \mapsto \{x_1, ..., x_n\}$. Give $\mathfrak{A}^o$ its relative product topology as a subspace of X^n. It is not hard to show that π is an $n!$-to-one covering map. In particular, the relative Vietoris topology on $\mathfrak{A}$ coincides with the quotient topology induced by π.

4.2 Topological orthoalgebras

The foregoing considerations lead to what seems a very natural notion of "topological quantum logics". In brief, given a topological *algebraic* test space $(X, \mathfrak{A})$, one can endow its space of events, $\mathcal{E}$, with a standard topology defined exactly as was done above for $\mathfrak{A}$. We can now topologize the logic $L = \Pi(X, \mathfrak{A})$ of a topological test by giving it the quotient topology induced by the canonical quotient map $A \mapsto p(A)$ sending each event A to its associated proposition ($\sim$-equivalence class). Under a few natural and not very restrictive assumptions, this topology articulates well with the orthoalgebraic structure of L.

4.6 Definition: A **topological orthoalgebra** is an orthoalgebra L, equipped with a topology with respect to which

(a) $\perp$ is closed in L^2;

(b) $\oplus : \perp \to L$ and $' : L \to L$ are both continuous.

4.7 Examples: A *topological OML* (TOML) [1] is an orthomodular lattice equipped with a topology making the operations $\vee : L \times L \to L$ and $' : L \to L$ continuous. It is easily checked that any TOML is a TOA. On the other hand, the lattice $L(\mathbf{H})$ of projections on a Hilbert space (in its norm topology) gives an example of a lattice-ordered TOA that is *not* a TOML. In view of this, the following is perhaps a little surprising [24]:

4.8 Theorem: *Let L be a compact Boolean TOA. Then L is a compact topological Boolean algebra.*

It is a standard result in the theory of topological lattices that a compact topological Boolean algebra must have the form 2^S, where 2 is the trivial two-element Boolean algebra (in its discrete topology), S is a set, and 2^S has its product topology. In particular, a compact TBA is complete and atomic. It follows that the *center* [7] of a compact TOA is a complete atomic Boolean algebra. If 0 is an isolated point, one can show that L has finite height [24]; accordingly, L's center is a finite Boolean algebra, and L therefore decomposes into a finite direct sum of simple compact TOAs.

If $(X, \mathfrak{A})$ is a topological algebraic test space, one can endow its space $\mathcal{E}(X, \mathfrak{A})$ of events with a topology in exactly the same manner as was done above for tests. One can then give the logic $\Pi(X, \mathfrak{A})$ the quotient topology induced by the canonical surjection $p : \mathcal{E} \to \Pi(X, \mathfrak{A})$. The question arises: when is Π, thus topologized, a topological orthoalgebra? In [25], it is established that this is the case if (i) the space $\mathcal{E}$ of events is closed in the Vietoris topology on 2^X (the hyperspace of all closed subsets of X, and (ii) for every open subset $\mathcal{U}$ of $\mathcal{E}$, the set $\mathcal{U}^{\mathsf{oc}}$ of events complementary to events in $\mathcal{U}$ is again open. A test space satisfying the latter condition is said to be *stably complemented.* The logic of any compact stably-complemented topological test space with a closed set of events provides an example of a compact TOA in which 0 is an isolated point.

5 Symmetry in Quantum Logic

In addition to having a richer topological character than their classical counterparts, quantum test spaces are much more highly *symmetrical* objects. If G is a group, let us agree that a *G-test space* [22] is a test space $(X, \mathfrak{A})$ where X is a G-space and where the action of G permutes the elements of $\mathfrak{A}$. In particular, this means that if $x \perp y$ in X, then $gx \perp gy$ for all $g \in G$. By a *topological* G-test space, we mean a G-test space $(X, \mathfrak{A})$ such $(X, \mathfrak{A})$ is a topological test space and the action of G on X is continuous. The following is easily proved:

5.1 Lemma: *Let $(X, \mathfrak{A})$ be a locally finite topological G-test space. Then the natural action of G on $\mathcal{E}(X, \mathfrak{A})$ is likewise continuous.*

Note that if G is compact and acts transitively on X, then X is also compact; hence, by Proposition 1, $(X, \mathfrak{A})$ will be of finite – hence, uniform – rank n.

5.2 Definition: We say that a G-test space $(X, \mathfrak{A})$ is *symmetric* iff (i) for every pair $E, F \in \mathfrak{A}$, there exists some $g \in G$ with $gE = F$, and (ii) for every $E \in \mathfrak{A}$ and every pair $x, y \in E$, there is some $g \in G$ with $gE = E$ and $gx = y$.

In other words, $(X, \mathfrak{A})$ is G-symmetric iff G acts transitively on $\mathfrak{A}$, and if the stabilizer of a test acts transitively on that test. By way of example, the quantum test space $(X(\mathbf{H}), \mathfrak{F}(\mathbf{H}))$ associated with a Hilbert space is $U(\mathbf{H})$-transitive, where $U(\mathbf{H})$ is the unitary group of $\mathbf{H}$.

The structure of any G-symmetric test space is entirely determined once we know the group G, the stabilizer $H := G_E$ of a test $E \in \mathfrak{A}$, and the stabilizer $K = G_x$ of an outcome belonging to E. Indeed, let G be any group, and H, K any two subgroups of G. Let G/K denote the space of left K-cosets of G. For any set $A \subseteq G$, let $A \cdot K := \{aK | a \in A\}$. We define

$$G/H; K := \{gH \cdot K | g \in G\}.$$

That is, $G/H; K$ is the set of all *sets* of left cosets of K of the form $\{ghK | h \in H\}$, as g ranges over G. It is easily checked that $(G/K, G/H; K))$ is a G-symmetric test space. Conversely, let $(X, \mathfrak{A})$ be a G-symmetric test space. Choose any outcome $x_o \in X$ and any test $E_o \in \mathfrak{A}$ with $x_o \in E$. Set $K = G_{x_o}$, the stabilizer of $x_o \in G$, and $H = G_{E_o}$. Let us agree to write x_g for gx_o and E_g for $gE_o = \{x_g | x \in E_o\}$. Then there is a natural bijection $gK \mapsto x_g$ from G/K to X making $(X, \mathfrak{A})$ isomorphic as a G-test space to $(G/K, G/H; K)$.

Evidently, then, we can read off all of the structure of a G-symmetric test space $(X, \mathfrak{A})$ from the group-theoretic data G, H, K. For instance, one has the following equivalences (all easily verified):

5.3 Lemma: *As above, let $x_o \in E_o \in \mathfrak{A}$, and set $x_a = ax_o$ and $E_a = aE_o$ for all $a \in G$. Then*

(a) $x_a \in E_b$ *iff* $b^{-1}a \in HK$

(b) $E_a \cap E_b \neq \emptyset$ *iff* $b^{-1}a \in HKH$

(c) $\{x_a | a \in A\}$ *is an event iff* $\bigcap_{a \in A} aKH \neq \emptyset$

(d) $x_a \perp x_b$ *iff* $b^{-1}a \in KHK \setminus K$.

Now suppose that G is a topological group. One would like to obtain a topological test space by identifying X with G/K in the latter's quotient topology. Where G is compact, this works out splendidly:

5.4 Theorem [25]**:** *Let G be a compact topological group and $H, K \leq G$ two closed subgroups. Let $(X, \mathfrak{A})$ be the associated symmetric G-test space, as described above. Give $X := G/K$ the quotient topology. Then $(X, \mathfrak{A})$ is a topological test space iff $H \setminus K$ is closed in G. In this case, the standard (Vietoris) topology on $\mathfrak{A}$ makes it homeomorphic to G/G_E, where G_E is the stabilizer in G of $H \cdot K$.*

As mentioned above, a quantum test space is symmetric with respect to the unitary group of the underlying Hilbert space. In fact, quantum test spaces satisfy an even stronger transitivity condition, which we abstract as follows:

5.5 Definition: We say that a G-test space $(X, \mathfrak{A})$ is **strongly G-symmetric** iff, for all test $E, F \in \mathfrak{A}$,

(a) $|E| = |F|$, and

(b) for every bijection $f : E \to F$, there exists a unique $g \in G$ with $gx = f(x)$ for every $x \in E$.

Since every bijection between two orthonormal bases for a Hilbert space extends to a unique unitary transformation, the quantum test space $(S_{\mathbf{H}}, \mathfrak{F}_{\mathbf{H}})$ is strongly symmetric under the unitary group of $\mathbf{H}$.

Notice that $(X, \mathfrak{A})$ is strongly G-symmetric if G acts principally on the associated space $\mathfrak{A}^o$ of ordered tests. That is, given two ordered tests $\vec{E} = (x_1, ..., x_n)$ and $\vec{F} = (y_1, ..., y_n)$, there is a *unique* group element $g \in G$ such that $\vec{F} = g\vec{F} = (gx_1, ..., gx_n)$. Now, any principal, Hausdorff G-space S for a compact group G is homeomorphic to G. Indeed, choose any "base-point" $\alpha \in S$: the mapping $g \mapsto g\alpha$ gives us a continuous bijection $G \to S$. If G is compact and S is Hausdorff, this is an homeomorphism. Thus we have the

5.6 Theorem: *Let G be a compact group, and let $(X, \mathfrak{A})$ be a strongly symmetric topological G-test space. Then $\mathfrak{A}$, in its Vietoris topology, is locally homeomorphic to G.*

Proof: In view of the preceding remark, it will be enough to show $\mathfrak{A}$ is locally homeomorphic to $\mathfrak{A}^o$. But since G is compact, so is X (the continuous image of G under the mapping $g \mapsto x_g := gx_o$). Thus, $\mathfrak{A}$ has finite rank. In this case, as observed above, the mapping $\mathfrak{A}^o \to \mathfrak{A}$ is an $n!$-to-one covering map, hence, a local homeomorphism. $\square$

Closing Remarks

We've considered a number of respects in which quantum test spaces are special: their logics are lattice-ordered (that is, OMLs), they have a natural and non-trivial topological structure, and they are strongly symmetric with respect to their unitary groups. Certainly these features alone do not pick out the quantum test spaces uniquely; however, they may well get us into the right arena. In any event, it seems to me very likely that *some* considerations of symmetry and topology will come into any account of quantum mechanics as a "law of thought"[f]

There is a great deal more to be done along the lines sketched here. A natural next step would be to try to determine the *local* structure of, say, rank 3 test spaces symmetric with respect to one of the classical Lie groups. Another worth-while project would be to develop a theory of tensor products for symmetric test spaces, building on what is already known about tensor products of algebraic test spaces generally. [5] [20].

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[f]as Chris Fuchs would put it.

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