COMPACT ORTHOALGEBRAS

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ABSTRACT. We initiate a study of topological orthoalgebras (TOAs), concentrating on the compact case. Examples of TOAs include topological orthomodular lattices, and also the projection lattice of a Hilbert space. As the latter example illustrates, a lattice-ordered TOA need not be a topological lattice. However, we show that a compact Boolean TOA is a topological Boolean algebra. Using this, we prove that any compact regular TOA is atomistic , and has a compact center. We prove also that any compact TOA with isolated 0 is of finite height. We then focus on *stably ordered* TOAs: those in which the upper-set generated by an open set is open. These include both topological orthomodular lattices and interval orthoalgebras – in particular, projection lattices. We show that the topology of a compact stably-ordered TOA with isolated 0 is determined by that of of its space of atoms.

1. Introduction

Broadly speaking, a *quantum logic* is any of a range of order-theoretic and partial-algebraic structures – orthomodular lattices and posets, orthoalgebras, and effect algebras – abstracted from the projection lattice $L(\mathbf{H})$ of a Hilbert space $\mathbf{H}$. Since the primordial example is very much a topological object, it would seem natural to undertake a study of "topological quantum logics" more generally. There does exist a literature devoted to topological orthomodular lattices (e.g., [3, 4, 12]); however $L(\mathbf{H})$, in its norm or strong operator topology, is not a topological lattice, the meet and join in $L(\mathbf{H})$ being not continuous. On the other hand, $L(\mathbf{H})$ *is* a topological orthoalgebra in a natural sense – as, indeed, are many other orthoalgebras one meets in practice, including all topological orthomodular lattices.

The purpose of this paper is to begin a systematic study of topological orthoalgebras (TOAs) *in abstracto*. In the interest of making what follows self-contained, section 2 collects some general background material on orthoalgebras. Section 3 develops some of the general theory of TOAs, with a focus on the compact case. Among other things, it is shown that a compact Boolean TOA is a topological Boolean algebra. This is a non-trivial fact, since, as the example of $L(\mathbf{H})$ shows, a lattice-ordered TOA need not be a topological lattice. We also show that any (algebraically) regular compact topological orthomodular poset is atomistic , and that a compact TOA with 0 isolated is atomistic and of finite height. In section 4, we consider a class of TOAs we call *stably ordered*: those in which the upper-set generated by an open set is again open. This includes all topological orthomodular lattices and also projection lattices. We show that the topology of a stably-ordered TOA with 0 isolated is entirely determined by that on its space of atoms.

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2. Background

If $(L, \leq, ', 0, 1)$ is any orthocomplemented poset, we call elements a and b of L *orthogonal*, writing $a \perp b$, iff $a \leq b'$. Suppose that any two orthogonal elements of L have a join. Then for $a \leq b$ in L, we can define a relative complement $b \wedge a' \perp a$. L is an *orthomodular poset* (hereafter: OMP) iff, in addition,

$$a \leq b \;\Rightarrow\; (b \wedge a') \vee a = b. \tag{2.1}$$

An *orthomodular lattice* (OML) is a lattice-ordered OMP. Evidently, the "orthomodular identity" (2.1) is a weak form of distributivity, and thus every Boolean algebra is an OML. The primordial (non-Boolean) example is the lattice $L(\mathbf{H})$ of projections – equivalently, closed subspaces – of a Hilbert space $\mathbf{H}$. Orthomodular lattices and posets have been studied extensively. The standard reference is [9]; for a more recent survey, see [2].

Let us agree to write $a \oplus b$ for the join of orthogonal elements a and b of an orthocomplemented poset, whenever this join exists. It is not difficult to check that L is an OMP iff the resulting structure $(L, \oplus)$ satisfies the conditions that (i) $a \oplus b$ exists whenever $a \perp b$, and (ii) if $a \oplus b = 1$, then $b = a'$. This suggests the following.

Definition 2.1. An *orthoalgebra* is a structure $(L, \oplus)$ consisting of a set L, an associative, commutative[1] partial binary operation $\oplus$ on L, such that for all $a \in L$

(a) there exists a unique element $a' \in L$ with $a \oplus a' = 1$;
(b) $a \oplus a$ exists only if $a = 1'$.

We write $a \perp b$ to indicate that $a \oplus b$ exists. Also, we write 0 for $1'$, noticing that $0 \oplus a = a$ for every $a \in L$. A set $S \subseteq L$ is a *sub-orthoalgebra* of L iff it contains 0 and is closed under both $'$ and $\oplus$ (whenever the latter is defined).

Orthoalgebras were introduced in the early 1980s by D. J. Foulis and C. H. Randall [6] in connection with the problem of defining tensor products of quantum logics. Further information can be found in [5] and [14]. For later reference, we mention that an *effect algebra* (see, e.g., [1]) is a structure $(L, \oplus)$, satisfying all of the conditions of Definition 2.1 save (b), which is replaced by the weaker condition that for all $a \in L$, if $a \perp 1$, then $a = 0$.

2.1. Orthoalgebras as orthoposets.

From the remarks preceding Definition 1.1, it is clear that any OMP gives rise to an orthoalgebra in which $a \oplus b = a \vee b$. Any orthoalgebra $(L, \oplus)$ can be partially ordered by setting $a \leq b$ iff there exists $c \in L$ with $b = a \oplus c$. The operation $a \mapsto a'$ is an orthocomplementation with respect to this ordering. Thus, any orthoalgebra gives rise to an orthoposet. Moreover, for any $a \leq b$ in L, there is a unique element $c \in L$ – namely, $(b' \oplus a)'$ – such that $b = a \oplus c$. It is usual to call this element $b \ominus a$. If L is an OMP, this is exactly $b \wedge a'$. In this language, the orthomodular law (1) becomes

$$a \leq b \;\Rightarrow\; b = (b \ominus a) \oplus a, \tag{2.2}$$

which holds in any orthoalgebra. In general, however, $a \oplus b$ is not the join, but only a *minimal* upper bound, for orthogonal elements a and b of an orthoalgebra L. Indeed, one can show that the orthoposet $(L, \leq, ', 0, 1)$ obtained from $(L, \oplus)$ is an OMP if and only if $a \oplus b = a \vee b$ for all $a, b \in L$; this in turn is equivalent to

[1] The associativity and commutativity of $\oplus$ are here to be understood in the strong sense, i.e., if $a \oplus b$ is defined, then so is $b \oplus a$, and the two are equal, and if $a \oplus (b \oplus c)$ is defined, so is $(a \oplus b) \oplus c$, and the two are equal.

the condition, called *orthocoherence* in the literature, that if $a, b, c \in L$ are pairwise orthogonal, then $a \perp (b \oplus c)$, so that $a \oplus (b \oplus c)$ exists. Thus, orthomodular posets are effectively the same thing as orthocoherent orthoalgebras, and orthomodular lattices are effectively the same things as lattice-ordered orthoalgebras.

2.2. **Boolean orthoalgebras and Compatibility.** An orthoalgebra $(L, \oplus)$ is said to be *Boolean* iff the corresponding orthoposet $(L, \leq, ', 0, 1)$ is a Boolean lattice. A subset of L is said to be *compatible* iff it is contained in a Boolean sub-orthoalgebra of L. Two elements $a, b \in L$ are compatible iff there exist elements a_1, b_1 and c with $a = a_1 \oplus c$, $b = c \oplus b_1$, and $a \perp b_1$, so that $a_1 \oplus c \oplus b_1$ exists [5]. Equivalently, a and b are compatible iff there exists an element $c \leq a, b$ with $a \perp (b \ominus c)$. The triple $(a_1, c, b_1) = (a \ominus c, c, b \ominus c)$ is then called a *Mackey decomposition* for a and b. If L is Boolean, then every pair of elements $a, b \in L$ has a *unique* Mackey decomposition, namely, $(a \ominus b, a \wedge b, b \ominus a)$. It is possible, even in an OMP, for a pairwise compatible set of elements not to be compatible. An orthoalgebra in which pairwise compatible sets *are* compatible is said to be *regular*. Note that such an orthoalgebra is automatically orthocoherent, i.e., an OMP.

2.3. **The center of an orthoalgebra.** For any element a of an orthoalgebra L, the interval $[0, a] = \{x \in L | 0 \leq x \leq a\}$ is itself an orthoalgebra, the orthogonal sum of $x, y \leq a$ being given by $x \oplus y$, provided this exists in L *and* is again below a. There is a natural mapping $[0, a] \times [0, a'] \to L$ given by $(x, y) \mapsto x \oplus y$. If this is an isomorphism, a is said to be *central*. The *center* of L is the set $\mathbf{C}(L)$ of all central elements of L. It can be shown [8] that $\mathbf{C}(L)$ is a Boolean sub-orthoalgebra of L. In particular, L is Boolean iff $L = \mathbf{C}(L)$. L is *irreducible* iff $\mathbf{C}(L) = \{0, 1\}$.

2.4. **Joint orthogonality.** A compatible pairwise-orthogonal set is said to be *jointly orthogonal*. Equivalently, $A \subseteq L$ is jointly orthogonal iff, for every finite subset $F = \{a_1, ..., a_n\} \subseteq A$, the "partial sum" $\bigoplus F = a_1 \oplus \cdots \oplus a_n$ exists. If the join of all partial sums of A exists, we denote it by $\bigoplus A$, and speak of this as the sum of A. We shall say that L is *orthocomplete* if every jointly orthogonal subset of L has a sum in this sense. An orthoalgebra is *atomistic* iff every element of L can be expressed as the sum of a jointly orthogonal set of atoms.

3. Topological Orthoalgebras

Definition 3.1. A *topological orthoalgebra* (hereafter: TOA) is an orthoalgebra $(L, \oplus)$ equipped with a topology making the relation $\perp \subseteq L \times L$ closed, and the mappings $\oplus : \perp \to L$ and $' : L \to L$, continuous.

One could define a topological effect algebra in just the same way. [2] We shall not pursue this further, except to note that the following would carry over verbatim to that context:

Lemma 3.2. *Let $(L, \oplus)$ be a topological orthoalgebra. Then*

(a) *The order relation $\leq$ is closed in $L \times L$*
(b) *L is a Hausdorff space.*
(c) *The mapping $\ominus : \leq \to L$ is continuous.*

[2] In a recent preprint [13], Riečanova offers a more general definition of topological effect algebras, not requiring the relation $\perp$ to be closed.

Proof. For (a), notice that $a \leq b$ iff $a \perp b'$. Thus, $\leq = f^{-1}(\perp)$ where $f : L \times L \to L \times L$ is the continuous mapping $f(a,b) = (a,b')$. Since $\perp$ is closed, so is $\leq$. That L is Hausdorff now follows by standard arguments (cf. [9, Ch. VII] or [12]). Finally, since $b \ominus a = (b' \oplus a)'$, and $\oplus$ and $'$ are both continuous, $\ominus$ is also continuous. □

3.1. **Examples.** Any product of discrete orthoalgebras, with the product topology, is a TOA. Another source of examples are topological orthomodular lattices (TOMLs) [3, 4]. A TOML is an orthomodular lattice equiped with a Hausdorff topology making both the lattice operations and the orthocomplementation continuous. If L is a TOML and $a, b \in L$, then $a \perp b$ iff $a \leq b'$ iff $a = a \wedge b'$. This is obviously a closed relation, since L is Hausdorff and both $\wedge$ and $'$ are continuous. Thus, every TOML may be regarded as a TOA. However, there are simple and important examples of lattice-ordered TOAs that are not TOMLs:

Example 3.3. L be the horizontal sum of four-element Boolean algebras $L_x = \{0, x, x', 1\}$ with x (and hence, x') parametrized by a non-degenerate real interval $[a, b]$. Topologize this as two disjoint copies of $[a, b]$ plus two isolated points 0 and 1. The orthogonality relation is then obviously closed, and $\oplus$ is obviously continuous; however, if we let $x \to x_o$ (with $x \neq x_o$) in I, then we have $x \wedge x_o = 0$ yet $x_o \wedge x_o = x_o$; hence, $\wedge$ is not continuous.

Example 3.4. Let $\mathbf{H}$ be a Hilbert space, and let $L = L(\mathbf{H})$ be the space of projection operators on $\mathbf{H}$, with its operator-norm topology. As multiplication is jointly continuous, the relation $P \perp Q$ iff $PQ = QP = 0$ is closed. Since addition and subtraction are continuous, the partial operation $P, Q \mapsto P \oplus Q := P + Q$ is continuous on $\perp$, as is the operation $P \mapsto P' := \mathbf{1} - P$. Thus, $L(\mathbf{H})$ is a lattice-ordered topological orthoalgebra. It is not, however, a topological lattice. Indeed, if Q is a non-trivial projection, choose unit vectors x_n not lying in $\mathrm{ran}(Q)$, but converging to a unit vector in $x \in \mathrm{ran}(Q)$. If P_n is the projection generated by x_n and P, that generated by x, then $P_n \to P$. But $P_n \wedge Q = 0$, while $P \wedge Q = P$.

We can also endow $L(\mathbf{H})$ with the relative strong operator topology, or, equivalently, the relative weak operator topology (these coinciding for projections). Multiplication is jointly SOT-continuous for operators of norm ≤ 1, the same arguments as given above show that this topology also makes $L(\mathbf{H})$ a lattice-ordered TOA, but not a topological lattice.

Remark 3.5. Topologically, projection lattices and TOMLs are strikingly different. Any compact TOML is totally disconnected [4, Lemma 3]. In strong contrast to this, if $\mathbf{H}$ is finite dimensional, then $L(\mathbf{H})$ is compact, but the set of projections of a given dimension in $L(\mathbf{H})$ is a manifold. As this illustrates, TOAs are much freer objects topologically than TOMLs. Indeed, by an easy generalization of Example 3.3, any Hausdorff space can be embedded in a TOA.

3.2. **Compact Orthoalgebras.** For the balance of this paper, we concentrate on compact TOAs. It is a standard fact [9, Corollary VII.1.3] that any ordered topological space with a closed order is isomorphic to a closed subspace of a cartesian power of the real unit interval $[0, 1]$ in its product order and topology. It follows that such a space L is *topologically order-complete*, meaning that any upwardly-directed net in L has a supremum, to which it converges. Applied to a compact TOA, this yields the following completeness result:

Lemma 3.6. *Any compact TOA L is orthocomplete. Moreover, if $A \subseteq L$ is jointly orthogonal, the net of finite partial sums of A converges topologically to $\bigoplus A$.*

We are going to show (Theorem 3.12) that any compact regular TOA is atomistic. In aid of this, the following technical definition proves most useful:

Definition 3.7. If L is any orthoalgebra, let

$$\mathbf{M}(L) := \{(a,c,b) \in L \times L \times L | c \leq a,\ c \leq b,\ \text{and}\ a \perp (b \ominus c)\}.$$

In other words, $(a,c,b) \in \mathbf{M}(L)$ iff $(a \ominus c, c, b \ominus c)$ is a Mackey decomposition for a and b.

Lemma 3.8. *For any TOA L, the relation $\mathbf{M}(L)$ is closed in $L \times L \times L$.*

Proof. Just note that $\mathbf{M}(L) = (\geq \times L) \cap (L \times \leq) \cap (\mathrm{Id} \times \ominus)^{-1}(\perp)$. Since the relations $\leq$ and $\perp$ are closed and $\ominus : \leq \to L$ is continuous, this also is closed. □

Since lattice-ordered TOAs need not be topological lattices, the following is noteworthy:

Proposition 3.9. *A compact Boolean topological orthoalgebra is a topological lattice, and hence, a compact topological Boolean algebra.*

Proof. If L is Boolean, then $\mathbf{M}(L)$ is, up to a permutation, the graph of the mapping $a, b \mapsto a \wedge b$. Thus, by Lemma 3.8, $\wedge$ has a closed graph. Since L is compact, this suffices to show that $\wedge$ is continuous.[3] It now follows from the continuity of $'$ that $\vee$ is also continuous. □

Note that every compact topological Boolean algebra has the form 2^E, where E is a set and 2^E has the product topology [8]. In particular, every compact Boolean algebra is atomistic . This will be useful below.

Question 3.10. Is *every* Boolean TOA a topological Boolean algebra?

For any orthoalgebra L, let $\mathrm{Comp}(L)$ be the set of all compatible pairs in L, and for any fixed $a \in L$, let $\mathrm{Comp}(a)$ be the set of elements compatible with a.

Proposition 3.11. *Let L be a compact TOA. Then*
- (a) *$Comp(L)$ is closed in $L \times L$;*
- (b) *For every $a \in L$, $Comp(a)$ is closed in L;*
- (c) *The closure of a pairwise compatible set in L is pairwise compatible;*
- (d) *A maximal pairwise compatible set in L is closed.*

Proof. (a) $\mathrm{Comp}(L) = (\pi_1 \times \pi_3)(\mathbf{M}(L))$. Since $\mathbf{M}(L)$ is closed, and hence compact, and $\pi_1 \times \pi_3$ is continuous, $\mathrm{Comp}(L)$ is also compact, hence closed. For (b), note that $\mathrm{Comp}(a) = \pi_1(\mathrm{Comp}(L) \cap (L \times \{a\}))$. As $\mathrm{Comp}(L)$ is closed, so is $\mathrm{Comp}(L) \cap (L \times \{a\})$; hence, its image under π_1 is also closed (remembering here that L is compact). For (c), suppose $M \subseteq L$ is pairwise compatible. Then $M \times M \subset \mathrm{Comp}(L)$. By part (a), $\mathrm{Comp}(L)$ is closed, so we have

$$\overline{M} \times \overline{M} \subseteq \overline{M \times M} \subseteq \mathrm{Comp}(L),$$

[3] Recall here that if X and Y are compact and the graph G_f of $f : X \to Y$ is closed, then f is continuous. Indeed, let $F \subseteq Y$ be closed. Then $f^{-1}(F) = \pi_1((X \times F) \cap G_f)$, where π_1 is projection on the first factor. Since X and Y are compact, π_1 sends closed sets to closed sets.

whence, $\overline{M}$ is again pairwise compatible. Finally, for (d), if M is a maximal pairwise compatible set, then the fact that $M \subseteq \overline{M}$ and $\overline{M}$ is also pairwise compatible entails that $M = \overline{M}$. □

There exist (non-orthocoherent) orthoalgebras in which $\text{Comp}(L) = L \times L$ ([5], Example 3.5). However, in an OML, $\text{Comp}(L) = \mathbf{C}(L)$, the center of L. Thus we recover from part (a) of Proposition 3.11 the fact (not hard to prove directly; see [3]) that the center of a compact TOML is a compact Boolean algebra.

In fact, we get a good deal more than this. Recall that an orthoalgebra regular iff every pairwise compatible subset is contained in a Boolean sub-orthoalgebra. Many orthoalgebras that arise in practice, including all lattice-ordered orthoalgebras, are regular. A *block* in an orthoalgebra is a maximal Boolean sub-orthoalgebra. In a regular orthoalgebra, this is the same thing as a maximal pairwise compatible set.

Theorem 3.12. *Let L be a compact, regular TOA. Then*

(a) *Every block of L is a compact Boolean algebra, as is the center of L;*
(b) *L is atomistic .*

Proof. (a) If L is regular, then a block of L is the same thing as a maximal pairwise compatible set. It follows from part (d) of Proposition 3.11 that every block is closed in L, and hence compact. It is not hard to show that in a regular TOA the center is the intersection of the blocks. Thus we also have that $\mathbf{C}(L)$ is also closed, hence compact. Proposition 3.9 now supplies the result.

To prove (b), suppose $a \in L$. By Zorn's Lemma, there is some block $B \subseteq L$ with $a \in B$. Since B is a compact Boolean algebra, it is complete and atomistic ; hence, a can be written as the join, $\bigvee_B A$, of a set A of atoms in B. Equivalently, $a = \bigvee_B \{\bigoplus F | F \subseteq A, F \text{ finite}\}$. By lemma 3.6, L is orthocomplete, hence, $\bigoplus A = \bigvee_L \{\bigoplus F | F \subseteq A,\ F \text{ finite}\}$ also exists, and is the limit of the partial sums $\bigoplus F$, $F \subseteq A$ finite. Since each partial sum lies in B, and B is closed, $\bigoplus A \in B$. It follows that $\bigoplus A = a$. It remains to show that every atom of B is an atom of L. Suppose that b is an atom of B and that $x \in L$ with $0 < x \leq b$. Since B is Boolean, every $y \in B$ satisfies either $b \leq y$ or $y \leq b'$; thus, either $x \leq y$ or $y \leq x$. In particular, x is compatible with every element of B. Since a block in a regular orthoalgebra is a maximal pairwise compatible set, $x \in B$, whence, $x = b$. □

3.3. **TOAs with Isolated Zero.** In [3], it is established that any TOML with an isolated point is discrete. In particular, a *compact* TOML with an isolated point is finite. This does not hold for lattice-ordered TOAs generally. Indeed, if $\mathbf{H}$ is a finite-dimensional Hilbert space, then $L(\mathbf{H})$ is a compact lattice-ordered TOA in which 0 is isolated. On the other hand, as we now show, a compact TOA with isolated zero does have quite special properties. We begin with an elementary but important observation. Call an open set in a TOA space *totally non-orthogonal* if it contains no two orthogonal elements.

Proposition 3.13. *Every non-zero element of a TOA has a totally non-orthogonal open neighborhood.*

Proof. Let L be a TOA. If $a \neq 0$, then $(a, a) \not\in \perp$. Since the latter is closed in L^2, we can find open sets U and V with $(a, a) \in U \times V$ and $(U \times V) \cap \perp = \emptyset$. The set $U \cap V$ is a totally non-orthogonal open neighborhood of a. □

Proposition 3.14. *Let L be a compact TOA with 0 isolated. Then*

(a) *L is atomistic and of finite height;*
(b) *The set of atoms of L is open.*

Proof. (a) We first show that there is a finite upper bound on the size of a pairwise orthogonal set. Since 0 is isolated in L, $L \setminus \{0\}$ is compact. By Lemma 3.13, we can cover $L \setminus \{0\}$ by finitely many totally non-orthogonal open sets $U_1, ..., U_n$. A pairwise-orthogonal subset of $L \setminus \{0\}$ can meet each U_i at most once, and so, can have at most n elements. Now given a finite chain $x_1 < x_2 < ... < x_m$ in L, we can construct a pairwise orthogonal set $y_1, ..., y_{m-1}$ defined by $y_1 = x_1$ and $y_k = x_k \ominus y_{k-1}$ for $k = 2, ..., m-1$. Hence, $m - 1 \leq n$, so $m \leq n + 1$. This shows that L has finite height, from which it follows that L is atomistic .

(b) Note that if A and B are any closed subsets of L, then $(A \times B) \cap \perp$ is a closed, hence compact, subset of $\perp$. Since $\oplus$ is continuous on $\perp$, the set

$$A \oplus B := \{a \oplus b | a \in A, b \in B \text{ and } a \perp b\} = \oplus((A \times B) \cap \perp)$$

is compact, hence closed. The set of non-atoms is precisely $(L \setminus \{0\}) \oplus (L \setminus \{0\})$. Since 0 is isolated, $(L \setminus \{0\})$ is closed. Thus, the set of non-atoms is closed. □

Remark 3.15. Notice that both the statements and the proofs of Lemma 3.13 and part (a) of Proposition 3.14 apply verbatim to any topological orthoposet, i.e., any ordered space having a closed order and equipped with a continuous orthocomplementation.

If a belongs to the center of a TOA L, then $[0, a] \times [0, a'] \subseteq \perp$. Hence, the natural isomorphism $\phi : [0, a] \times [0, a'] \to L$ given by $(x, y) \mapsto x \oplus y$ is continuous. If L is compact, then so are $[0, a]$ and $[0, a']$; hence, ϕ is also an homeomorphism. Since the center of an orthoalgebra is a Boolean sub-orthoalgebra of L, and since a Boolean algebra of finite height is finite, Proposition 3.14 has the following

Corollary 3.16. *Let L be a compact TOA with 0 isolated. Then the center of L is finite. In particular, L decomposes, both algebraically and topologically, as the product of finitely many compact irreducible TOAs.*

4. Stably Ordered Topological Orthoalgebras

In this section we consider a particularly tractable, but still quite broad, class of TOAs.

Definition 4.1. We shall call an ordered topological space L *stably ordered* iff, for every open set $U \subseteq L$, the upper-set $U\uparrow = \{b \in L | \exists a \in U a \leq b\}$ is again open.[4]

Remark 4.2. Note that this is equivalent to saying that the second projection mapping $\pi_2 : \leq \to L$ is an open mapping, since for open sets $U, V \subseteq L$,

$$\pi_2((U \times V) \cap \leq) = U\uparrow \cap V.$$

Note, too, that if L carries a continuous orthocomplementation $'$, then L is stably ordered iff $U\downarrow = \{x | \exists y \in U, x \leq y\}$ is open for all open sets $U \subseteq L$.

[4]The term used by Priestley [13] is "space of type I_i."

Example 4.3. The following example (a variant of Example 3.3) shows that a TOA need not be stably ordered. Let $L = [0, 1/4] \cup [3/4, 1]$ with its usual topology, but without its usual order. For $x, y \in L$, set $x \perp y$ iff $x + y = 1$ or $x = 0$ or $y = 0$. In any of these cases, define $x \oplus y = x + y$. As is easily checked, this is a compact lattice-ordered TOA. However, for the clopen set $[0, 1/4]$ we have $[0, 1/4]\!\uparrow = [0, 1/4] \cup \{1\}$, which is certainly not open.

Such examples notwithstanding, most of the orthoalgebras that arise "in nature" do seem to be stably ordered. The following is mentioned (without proof) in [13]:

Lemma 4.4. *Any topological $\wedge$-semilattice – in particular, any topological lattice – is stably ordered.*

Proof. If L is a topological meet-semilattice and $U \subseteq L$ is open, then

$$U\!\uparrow = \{\ x \in L \mid \exists y \in U\ x \wedge y \in U\} = \pi_1(\wedge^{-1}(U))$$

where $\pi_1 : L \times L \to L$ is the (open) projection map on the first factor and $\wedge : L \times L \to L$ is the (continuous) meet operation. □

Many orthoalgebras, including projection lattices, can be embedded in ordered abelian groups [1]. Indeed, suppose G is an ordered abelian group. If $e > 0$ in G, let $[0, e]$ denote the set of all elements $x \in G$ with $0 \leq x \leq e$. We can endow $[0, e]$ with the following partial-algebraic structure: for $x, y \in [0, e]$, set $x \perp y$ iff $x + y \leq e$, in which case let $x \oplus y = x + y$. Define $x' = e - x$. Then $([0, e], \oplus, ', 0, e)$ is an effect algebra. By a *faithful sub-effect algebra* of $[0, e]$, we mean a subset L of $[0, e]$, containing 0 and e, closed under both $\oplus$, where this is defined, and under $'$, *and* such that, for all $x, y \in L$, $x \leq y$ in G iff $\exists z \in L$ with $y = x + z$ (so that the order inherited from G and the order induced by $\oplus$ coincide.)

By way of example, let $L = L(\mathbf{H})$, the projection lattice of a Hilbert space $\mathbf{H}$, regarded as an orthoalgebra, and let $G = \mathcal{B}_{sa}(\mathbf{H})$, the ring of bounded self-adjoint operators on $\mathbf{H}$ with its operator-norm topology, ordered in the usual way. Then L is a faithful sub-effect algebra of $[0, \mathbf{1}]$, where $\mathbf{1}$ is the identity operator on $\mathbf{H}$. (This follows from the fact that, for projections $P, Q \in L(\mathbf{H})$, $P + Q \leq \mathbf{1}$ iff $P \perp Q$, and the fact that if $P \leq Q$ as positive operators, then $Q - P$ is a projection.)

Lemma 4.5. *Let L be an orthoalgebra, let G be any ordered topological abelian group with a closed cone (equivalently, a closed order), and suppose that L can be embedded as a sub-effect algebra of $[0, e]$, where $e > 0$ in G. Then L, in the topology inherited from G, is a stably ordered TOA.*

Proof. We may assume that L is a subspace of $[0, e]$. Since $x \perp y$ in L iff $x + y \leq e$, we have $\perp = +^{-1}([0, e]) \cap L$, which is relatively closed in L. The continuity of $\oplus$ and $'$ are automatic. Suppose now that $U \cap L$ is a relatively open subset of L. Then, since L is a faithful sub-effect algebra of $[0, e]$, the upper set generated by $U \cap L$ in L is $U\!\uparrow \cap L$, where $U\!\uparrow$ is the upper set of U in $[0, e]$. It suffices to show that this last is open. But $U\!\uparrow = \bigcup_{y \geq 0} U + y$, which is certainly open. □

In particular, it follows that the projection lattice $L(\mathbf{H})$ of a Hilbert space $\mathbf{H}$ is stably ordered in its norm topology.

Example 4.6. A *state* on an orthoalgebra $(L, \oplus)$ is a mapping $f : L \to [0, 1]$ such that $f(1) = 1$ and, for all $a, b \in L$, $f(a \oplus b) = f(a) \oplus f(b)$ whenever $a \oplus b$ exists. A set Δ of states on L is said to be *order-determining* iff $f(p) \leq f(q)$ for all $f \in \Delta$

implies $p \leq q$ in L. In this case the mapping $L \to \mathbb{R}^\Delta$ given by $p \mapsto \hat{p}$, where $\hat{p}(f) = f(p)$, embeds L as a faithful sub-effect algebra of the ordered abelian group $\mathbb{R}^\Delta$ (with pointwise addition, pointwise order). Since the positive cone of $\mathbb{R}^\Delta$ is closed in the product topology, Lemma 4.5 tells us that L is stably ordered in the topology inherited from pointwise convergence on Δ.

As a special case, the projection lattice $L = L(\mathbf{H})$ has an order-determining set of states of the form $f(P) = \langle Px, x\rangle$, where x is a unit vector in $\mathbf{H}$. If $\mathbf{H}$ is complex, the polarization identity tells us that the weakest topology on $L(\mathbf{H})$ making every such state continuous is exactly the restriction to $L(\mathbf{H})$ of the weak operator topology. Recall that this coincides with the strong operator topology on $L(\mathbf{H})$, and makes $L(\mathbf{H})$ a TOA. Thus, for a complex Hilbert space, $L(\mathbf{H})$ is stably-ordered in its strong operator topology.

If U and V are subsets of an orthoalgebra L, let us write $U \oplus V$ for $\oplus((U \times V) \cap \perp)$, i.e., for the set of all (existing) orthogonal sums $a \oplus b$ with $a \in U$ and $b \in V$.

Lemma 4.7. *A TOA L is stably ordered if, and only if, for every pair of open sets $U, V \subseteq L$, the set $U \oplus V$ is also open.*

Proof. Let U and V be any two open sets in L. Then

$$\begin{aligned} U \oplus V &= \{c \in L | c = a \oplus b, a \in U, b \in V\} \\ &= \{c \in L | \exists a \in U \ a \leq c \text{ and } c \ominus a \in V\} \\ &= \pi_2(\ominus^{-1}(V) \cap (L \times U\!\uparrow)). \end{aligned}$$

Now, since L is stably ordered, $U\!\uparrow$ is open, and hence, $\ominus^{-1}(V) \cap (L \times U\!\uparrow)$ is relatively open in $\leq$. But as observed above, for L stably ordered, $\pi_2 : \leq \to L$ is an open mapping, so $U \oplus V$ is open. For the converse, just note that $U\!\uparrow = U \oplus L$. □

Proposition 3.12 tells us that a compact TOA L with 0 isolated is atomistic and of finite height. It follows easily that every element of L can be expressed as a finite orthogonal sum of atoms. Let the *dimension*, $\dim(a)$, of an element $a \in L$ be the minimum number n of atoms $x_1, ..., x_n$ such that $a = x_1 \oplus \cdots \oplus x_n$. Note that $a \in L$ is an atom iff $\dim(a) = 1$.

Theorem 4.8. *Let L be a compact, stably-ordered TOA in which* 0 *is an isolated point. Then*

(a) *The set of elements of L of a given dimension is clopen.*
(b) *The topology on L is completely determined by that on the set of atoms.*

Proof. We begin by noting that if A and B are clopen subsets of L, then $A \oplus B$ is again clopen (open, by Lemma 4.7, and closed, because the image of the compact set $(A \times B) \cap \perp$ under the continuous map $\oplus$). Now, since 0 is isolated, $L \setminus \{0\}$ is clopen. Since the set of non-atoms in L is exactly $(L \setminus \{0\}) \oplus (L \setminus \{0\})$, it follows that the set of atoms is clopen. Define a sequence of sets L_k, $k \in \mathbb{N}$, by setting by $L_0 = \{0\}$, $L_1 =$ the set of atoms of L, and $L_{k+1} := L_k \oplus L_1$. These sets are clopen, as are all Boolean combinations of them. Thus, $\{a \in L | \dim(a) = k\} = L_k \setminus (\bigcup_{i=0}^{k-1} L_k)$ is clopen for every $k = 0, ..., \dim(L)$. This proves (a). For (b), it now suffices to show that the topology on each L_k is determined by that on L_1. Since L_1 and L_2 are clopen, Lemma 4.7 tells us that the mapping $\oplus : ((L_k \times L_1) \cap \perp) \to L_{k+1}$ is an open surjection, and hence, a quotient mapping. Thus, the topology on L_{k+1} is entirely determined by that on L_k and that on L_1. □

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