

# COMPACT ORTHOALGEBRAS

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ABSTRACT. We initiate a study of topological orthoalgebras (TOAs), concentrating on the compact case. Examples of TOAs include topological orthomodular lattices, and also the projection lattice of a Hilbert space. As the latter example illustrates, a lattice-ordered TOA need not be a topological lattice. However, we show that a compact Boolean TOA is a topological Boolean algebra. Using this, we prove that any compact regular TOA is atomistic, and has a compact center. We prove also that any compact TOA with isolated 0 is of finite height. We then focus on *stably ordered* TOAs: those in which the upper-set generated by an open set is open. These include both topological orthomodular lattices and interval orthoalgebras – in particular, projection lattices. We show that the topology of a compact stably-ordered TOA with isolated 0 is determined by that of its space of atoms.

## 1. INTRODUCTION

Broadly speaking, a *quantum logic* is any of a range of order-theoretic and partial-algebraic structures – orthomodular lattices and posets, orthoalgebras, and effect algebras – abstracted from the projection lattice  $L(\mathbf{H})$  of a Hilbert space  $\mathbf{H}$ . Since the primordial example is very much a topological object, it would seem natural to undertake a study of “topological quantum logics” more generally. There does exist a literature devoted to topological orthomodular lattices (e.g., [3, 4, 12]); however  $L(\mathbf{H})$ , in its norm or strong operator topology, is not a topological lattice, the meet and join in  $L(\mathbf{H})$  being not continuous. On the other hand,  $L(\mathbf{H})$  is a topological orthoalgebra in a natural sense – as, indeed, are many other orthoalgebras one meets in practice, including all topological orthomodular lattices.

The purpose of this paper is to begin a systematic study of topological orthoalgebras (TOAs) *in abstracto*. In the interest of making what follows self-contained, section 2 collects some general background material on orthoalgebras. Section 3 develops some of the general theory of TOAs, with a focus on the compact case. Among other things, it is shown that a compact Boolean TOA is a topological Boolean algebra. This is a non-trivial fact, since, as the example of  $L(\mathbf{H})$  shows, a lattice-ordered TOA need not be a topological lattice. We also show that any (algebraically) regular compact topological orthomodular poset is atomistic, and that a compact TOA with 0 isolated is atomistic and of finite height. In section 4, we consider a class of TOAs we call *stably ordered*: those in which the upper-set generated by an open set is again open. This includes all topological orthomodular lattices and also projection lattices. We show that the topology of a stably-ordered TOA with 0 isolated is entirely determined by that on its space of atoms.

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## 2. BACKGROUND

If  $(L, \leq', 0, 1)$  is any orthocomplemented poset, we call elements  $a$  and  $b$  of  $L$  *orthogonal*, writing  $a \perp b$ , iff  $a \leq b'$ . Suppose that any two orthogonal elements of  $L$  have a join. Then for  $a \leq b$  in  $L$ , we can define a relative complement  $b \wedge a' \perp a$ .  $L$  is an *orthomodular poset* (hereafter: OMP) iff, in addition,

$$(2.1) \quad a \leq b \Rightarrow (b \wedge a') \vee a = b.$$

An *orthomodular lattice* (OML) is a lattice-ordered OMP. Evidently, the “orthomodular identity” (2.1) is a weak form of distributivity, and thus every Boolean algebra is an OML. The primordial (non-Boolean) example is the lattice  $L(\mathbf{H})$  of projections – equivalently, closed subspaces – of a Hilbert space  $\mathbf{H}$ . Orthomodular lattices and posets have been studied extensively. The standard reference is [9]; for a more recent survey, see [2].

Let us agree to write  $a \oplus b$  for the join of orthogonal elements  $a$  and  $b$  of an orthocomplemented poset, whenever this join exists. It is not difficult to check that  $L$  is an OMP iff the resulting structure  $(L, \oplus)$  satisfies the conditions that (i)  $a \oplus b$  exists whenever  $a \perp b$ , and (ii) if  $a \oplus b = 1$ , then  $b = a'$ . This suggests the following.

**Definition 2.1.** An *orthoalgebra* is a structure  $(L, \oplus)$  consisting of a set  $L$ , an associative, commutative<sup>1</sup> partial binary operation  $\oplus$  on  $L$ , such that for all  $a \in L$

- (a) there exists a unique element  $a' \in L$  with  $a \oplus a' = 1$ ;
- (b)  $a \oplus a$  exists only if  $a = 1'$ .

We write  $a \perp b$  to indicate that  $a \oplus b$  exists. Also, we write  $0$  for  $1'$ , noticing that  $0 \oplus a = a$  for every  $a \in L$ . A set  $S \subseteq L$  is a *sub-orthoalgebra* of  $L$  iff it contains  $0$  and is closed under both  $'$  and  $\oplus$  (whenever the latter is defined).

Orthoalgebras were introduced in the early 1980s by D. J. Foulis and C. H. Randall [6] in connection with the problem of defining tensor products of quantum logics. Further information can be found in [5] and [14]. For later reference, we mention that an *effect algebra* (see, e.g., [1]) is a structure  $(L, \oplus)$ , satisfying all of the conditions of Definition 2.1 save (b), which is replaced by the weaker condition that for all  $a \in L$ , if  $a \perp 1$ , then  $a = 0$ .

**2.1. Orthoalgebras as orthoposets.** From the remarks preceding Definition 1.1, it is clear that any OMP gives rise to an orthoalgebra in which  $a \oplus b = a \vee b$ . Any orthoalgebra  $(L, \oplus)$  can be partially ordered by setting  $a \leq b$  iff there exists  $c \in L$  with  $b = a \oplus c$ . The operation  $a \mapsto a'$  is an orthocomplementation with respect to this ordering. Thus, any orthoalgebra gives rise to an orthoposet. Moreover, for any  $a \leq b$  in  $L$ , there is a unique element  $c \in L$  – namely,  $(b' \oplus a)'$  – such that  $b = a \oplus c$ . It is usual to call this element  $b \ominus a$ . If  $L$  is an OMP, this is exactly  $b \wedge a'$ . In this language, the orthomodular law (1) becomes

$$(2.2) \quad a \leq b \Rightarrow b = (b \ominus a) \oplus a,$$

which holds in any orthoalgebra. In general, however,  $a \oplus b$  is not the join, but only a *minimal* upper bound, for orthogonal elements  $a$  and  $b$  of an orthoalgebra  $L$ . Indeed, one can show that the orthoposet  $(L, \leq', 0, 1)$  obtained from  $(L, \oplus)$  is an OMP if and only if  $a \oplus b = a \vee b$  for all  $a, b \in L$ ; this in turn is equivalent to

<sup>1</sup>The associativity and commutativity of  $\oplus$  are here to be understood in the strong sense, i.e., if  $a \oplus b$  is defined, then so is  $b \oplus a$ , and the two are equal, and if  $a \oplus (b \oplus c)$  is defined, so is  $(a \oplus b) \oplus c$ , and the two are equal.

the condition, called *orthocoherence* in the literature, that if  $a, b, c \in L$  are pairwise orthogonal, then  $a \perp (b \oplus c)$ , so that  $a \oplus (b \oplus c)$  exists. Thus, orthomodular posets are effectively the same thing as orthocoherent orthoalgebras, and orthomodular lattices are effectively the same things as lattice-ordered orthoalgebras.

**2.2. Boolean orthoalgebras and Compatibility.** An orthoalgebra  $(L, \oplus)$  is said to be *Boolean* iff the corresponding orthoposet  $(L, \leq, ', 0, 1)$  is a Boolean lattice. A subset of  $L$  is said to be *compatible* iff it is contained in a Boolean sub-orthoalgebra of  $L$ . Two elements  $a, b \in L$  are compatible iff there exist elements  $a_1, b_1$  and  $c$  with  $a = a_1 \oplus c$ ,  $b = c \oplus b_1$ , and  $a \perp b_1$ , so that  $a_1 \oplus c \oplus b_1$  exists [5]. Equivalently,  $a$  and  $b$  are compatible iff there exists an element  $c \leq a, b$  with  $a \perp (b \ominus c)$ . The triple  $(a_1, c, b_1) = (a \ominus c, c, b \ominus c)$  is then called a *Mackey decomposition* for  $a$  and  $b$ . If  $L$  is Boolean, then every pair of elements  $a, b \in L$  has a *unique* Mackey decomposition, namely,  $(a \ominus b, a \wedge b, b \ominus a)$ . It is possible, even in an OMP, for a pairwise compatible set of elements not to be compatible. An orthoalgebra in which pairwise compatible sets *are* compatible is said to be *regular*. Note that such an orthoalgebra is automatically orthocoherent, i.e., an OMP.

**2.3. The center of an orthoalgebra.** For any element  $a$  of an orthoalgebra  $L$ , the interval  $[0, a] = \{x \in L \mid 0 \leq x \leq a\}$  is itself an orthoalgebra, the orthogonal sum of  $x, y \leq a$  being given by  $x \oplus y$ , provided this exists in  $L$  and is again below  $a$ . There is a natural mapping  $[0, a] \times [0, a'] \rightarrow L$  given by  $(x, y) \mapsto x \oplus y$ . If this is an isomorphism,  $a$  is said to be *central*. The *center* of  $L$  is the set  $\mathbf{C}(L)$  of all central elements of  $L$ . It can be shown [8] that  $\mathbf{C}(L)$  is a Boolean sub-orthoalgebra of  $L$ . In particular,  $L$  is Boolean iff  $L = \mathbf{C}(L)$ .  $L$  is *irreducible* iff  $\mathbf{C}(L) = \{0, 1\}$ .

**2.4. Joint orthogonality.** A compatible pairwise-orthogonal set is said to be *jointly orthogonal*. Equivalently,  $A \subseteq L$  is jointly orthogonal iff, for every finite subset  $F = \{a_1, \dots, a_n\} \subseteq A$ , the ‘‘partial sum’’  $\bigoplus F = a_1 \oplus \dots \oplus a_n$  exists. If the join of all partial sums of  $A$  exists, we denote it by  $\bigoplus A$ , and speak of this as the sum of  $A$ . We shall say that  $L$  is *orthocomplete* if every jointly orthogonal subset of  $L$  has a sum in this sense. An orthoalgebra is *atomistic* iff every element of  $L$  can be expressed as the sum of a jointly orthogonal set of atoms.

### 3. TOPOLOGICAL ORTHOALGEBRAS

**Definition 3.1.** A *topological orthoalgebra* (hereafter: TOA) is an orthoalgebra  $(L, \oplus)$  equipped with a topology making the relation  $\perp \subseteq L \times L$  closed, and the mappings  $\oplus : \perp \rightarrow L$  and  $' : L \rightarrow L$ , continuous.

One could define a topological effect algebra in just the same way.<sup>2</sup> We shall not pursue this further, except to note that the following would carry over verbatim to that context:

**Lemma 3.2.** *Let  $(L, \oplus)$  be a topological orthoalgebra. Then*

- (a) *The order relation  $\leq$  is closed in  $L \times L$*
- (b)  *$L$  is a Hausdorff space.*
- (c) *The mapping  $\ominus : \leq \rightarrow L$  is continuous.*

<sup>2</sup>In a recent preprint [13], Riečanová offers a more general definition of topological effect algebras, not requiring the relation  $\perp$  to be closed.

*Proof.* For (a), notice that  $a \leq b$  iff  $a \perp b'$ . Thus,  $\leq = f^{-1}(\perp)$  where  $f : L \times L \rightarrow L \times L$  is the continuous mapping  $f(a, b) = (a, b')$ . Since  $\perp$  is closed, so is  $\leq$ . That  $L$  is Hausdorff now follows by standard arguments (cf. [9, Ch. VII] or [12]). Finally, since  $b \ominus a = (b' \oplus a)'$ , and  $\oplus$  and  $'$  are both continuous,  $\ominus$  is also continuous.  $\square$

**3.1. Examples.** Any product of discrete orthoalgebras, with the product topology, is a TOA. Another source of examples are topological orthomodular lattices (TOMLs) [3, 4]. A TOML is an orthomodular lattice equipped with a Hausdorff topology making both the lattice operations and the orthocomplementation continuous. If  $L$  is a TOML and  $a, b \in L$ , then  $a \perp b$  iff  $a \leq b'$  iff  $a = a \wedge b'$ . This is obviously a closed relation, since  $L$  is Hausdorff and both  $\wedge$  and  $'$  are continuous. Thus, every TOML may be regarded as a TOA. However, there are simple and important examples of lattice-ordered TOAs that are not TOMLs:

**Example 3.3.**  $L$  be the horizontal sum of four-element Boolean algebras  $L_x = \{0, x, x', 1\}$  with  $x$  (and hence,  $x'$ ) parametrized by a non-degenerate real interval  $[a, b]$ . Topologize this as two disjoint copies of  $[a, b]$  plus two isolated points 0 and 1. The orthogonality relation is then obviously closed, and  $\oplus$  is obviously continuous; however, if we let  $x \rightarrow x_o$  (with  $x \neq x_o$ ) in  $I$ , then we have  $x \wedge x_o = 0$  yet  $x_o \wedge x_o = x_o$ ; hence,  $\wedge$  is not continuous.

**Example 3.4.** Let  $\mathbf{H}$  be a Hilbert space, and let  $L = L(\mathbf{H})$  be the space of projection operators on  $\mathbf{H}$ , with its operator-norm topology. As multiplication is jointly continuous, the relation  $P \perp Q$  iff  $PQ = QP = 0$  is closed. Since addition and subtraction are continuous, the partial operation  $P, Q \mapsto P \oplus Q := P + Q$  is continuous on  $\perp$ , as is the operation  $P \mapsto P' := \mathbf{1} - P$ . Thus,  $L(\mathbf{H})$  is a lattice-ordered topological orthoalgebra. It is not, however, a topological lattice. Indeed, if  $Q$  is a non-trivial projection, choose unit vectors  $x_n$  not lying in  $\text{ran}(Q)$ , but converging to a unit vector in  $x \in \text{ran}(Q)$ . If  $P_n$  is the projection generated by  $x_n$  and  $P$ , that generated by  $x$ , then  $P_n \rightarrow P$ . But  $P_n \wedge Q = 0$ , while  $P \wedge Q = P$ .

We can also endow  $L(\mathbf{H})$  with the relative strong operator topology, or, equivalently, the relative weak operator topology (these coinciding for projections). Multiplication is jointly SOT-continuous for operators of norm  $\leq 1$ , the same arguments as given above show that this topology also makes  $L(\mathbf{H})$  a lattice-ordered TOA, but not a topological lattice.

*Remark 3.5.* Topologically, projection lattices and TOMLs are strikingly different. Any compact TOML is totally disconnected [4, Lemma 3]. In strong contrast to this, if  $\mathbf{H}$  is finite dimensional, then  $L(\mathbf{H})$  is compact, but the set of projections of a given dimension in  $L(\mathbf{H})$  is a manifold. As this illustrates, TOAs are much freer objects topologically than TOMLs. Indeed, by an easy generalization of Example 3.3, any Hausdorff space can be embedded in a TOA.

**3.2. Compact Orthoalgebras.** For the balance of this paper, we concentrate on compact TOAs. It is a standard fact [9, Corollary VII.1.3] that any ordered topological space with a closed order is isomorphic to a closed subspace of a cartesian power of the real unit interval  $[0, 1]$  in its product order and topology. It follows that such a space  $L$  is *topologically order-complete*, meaning that any upwardly-directed net in  $L$  has a supremum, to which it converges. Applied to a compact TOA, this yields the following completeness result:

**Lemma 3.6.** *Any compact TOA  $L$  is orthocomplete. Moreover, if  $A \subseteq L$  is jointly orthogonal, the net of finite partial sums of  $A$  converges topologically to  $\bigoplus A$ .*

We are going to show (Theorem 3.12) that any compact regular TOA is atomistic. In aid of this, the following technical definition proves most useful:

**Definition 3.7.** If  $L$  is any orthoalgebra, let

$$\mathbf{M}(L) := \{(a, c, b) \in L \times L \times L \mid c \leq a, c \leq b, \text{ and } a \perp (b \oplus c)\}.$$

In other words,  $(a, c, b) \in \mathbf{M}(L)$  iff  $(a \oplus c, c, b \oplus c)$  is a Mackey decomposition for  $a$  and  $b$ .

**Lemma 3.8.** *For any TOA  $L$ , the relation  $\mathbf{M}(L)$  is closed in  $L \times L \times L$ .*

*Proof.* Just note that  $\mathbf{M}(L) = (\geq \times L) \cap (L \times \leq) \cap (\text{Id} \times \oplus)^{-1}(\perp)$ . Since the relations  $\leq$  and  $\perp$  are closed and  $\oplus : L \times L \rightarrow L$  is continuous, this also is closed.  $\square$

Since lattice-ordered TOAs need not be topological lattices, the following is noteworthy:

**Proposition 3.9.** *A compact Boolean topological orthoalgebra is a topological lattice, and hence, a compact topological Boolean algebra.*

*Proof.* If  $L$  is Boolean, then  $\mathbf{M}(L)$  is, up to a permutation, the graph of the mapping  $a, b \mapsto a \wedge b$ . Thus, by Lemma 3.8,  $\wedge$  has a closed graph. Since  $L$  is compact, this suffices to show that  $\wedge$  is continuous.<sup>3</sup> It now follows from the continuity of  $'$  that  $\vee$  is also continuous.  $\square$

Note that every compact topological Boolean algebra has the form  $2^E$ , where  $E$  is a set and  $2^E$  has the product topology [8]. In particular, every compact Boolean algebra is atomistic. This will be useful below.

*Question 3.10.* Is every Boolean TOA a topological Boolean algebra?

For any orthoalgebra  $L$ , let  $\text{Comp}(L)$  be the set of all compatible pairs in  $L$ , and for any fixed  $a \in L$ , let  $\text{Comp}(a)$  be the set of elements compatible with  $a$ .

**Proposition 3.11.** *Let  $L$  be a compact TOA. Then*

- (a)  *$\text{Comp}(L)$  is closed in  $L \times L$ ;*
- (b) *For every  $a \in L$ ,  $\text{Comp}(a)$  is closed in  $L$ ;*
- (c) *The closure of a pairwise compatible set in  $L$  is pairwise compatible;*
- (d) *A maximal pairwise compatible set in  $L$  is closed.*

*Proof.* (a)  $\text{Comp}(L) = (\pi_1 \times \pi_3)(\mathbf{M}(L))$ . Since  $\mathbf{M}(L)$  is closed, and hence compact, and  $\pi_1 \times \pi_3$  is continuous,  $\text{Comp}(L)$  is also compact, hence closed. For (b), note that  $\text{Comp}(a) = \pi_1(\text{Comp}(L) \cap (L \times \{a\}))$ . As  $\text{Comp}(L)$  is closed, so is  $\text{Comp}(L) \cap (L \times \{a\})$ ; hence, its image under  $\pi_1$  is also closed (remembering here that  $L$  is compact). For (c), suppose  $M \subseteq L$  is pairwise compatible. Then  $M \times M \subset \text{Comp}(L)$ . By part (a),  $\text{Comp}(L)$  is closed, so we have

$$\overline{M \times M} \subseteq \overline{\text{Comp}(L)} \subseteq \text{Comp}(L),$$

<sup>3</sup>Recall here that if  $X$  and  $Y$  are compact and the graph  $G_f$  of  $f : X \rightarrow Y$  is closed, then  $f$  is continuous. Indeed, let  $F \subseteq Y$  be closed. Then  $f^{-1}(F) = \pi_1((X \times F) \cap G_f)$ , where  $\pi_1$  is projection on the first factor. Since  $X$  and  $Y$  are compact,  $\pi_1$  sends closed sets to closed sets.

whence,  $\overline{M}$  is again pairwise compatible. Finally, for (d), if  $M$  is a maximal pairwise compatible set, then the fact that  $M \subseteq \overline{M}$  and  $\overline{M}$  is also pairwise compatible entails that  $M = \overline{M}$ .  $\square$

There exist (non-orthochoherent) orthoalgebras in which  $\text{Comp}(L) = L \times L$  ([5], Example 3.5). However, in an OML,  $\text{Comp}(L) = \mathbf{C}(L)$ , the center of  $L$ . Thus we recover from part (a) of Proposition 3.11 the fact (not hard to prove directly; see [3]) that the center of a compact TOML is a compact Boolean algebra.

In fact, we get a good deal more than this. Recall that an orthoalgebra regular iff every pairwise compatible subset is contained in a Boolean sub-orthoalgebra. Many orthoalgebras that arise in practice, including all lattice-ordered orthoalgebras, are regular. A *block* in an orthoalgebra is a maximal Boolean sub-orthoalgebra. In a regular orthoalgebra, this is the same thing as a maximal pairwise compatible set.

**Theorem 3.12.** *Let  $L$  be a compact, regular TOA. Then*

- (a) *Every block of  $L$  is a compact Boolean algebra, as is the center of  $L$ ;*
- (b)  *$L$  is atomistic .*

*Proof.* (a) If  $L$  is regular, then a block of  $L$  is the same thing as a maximal pairwise compatible set. It follows from part (d) of Proposition 3.11 that every block is closed in  $L$ , and hence compact. It is not hard to show that in a regular TOA the center is the intersection of the blocks. Thus we also have that  $\mathbf{C}(L)$  is also closed, hence compact. Proposition 3.9 now supplies the result.

To prove (b), suppose  $a \in L$ . By Zorn's Lemma, there is some block  $B \subseteq L$  with  $a \in B$ . Since  $B$  is a compact Boolean algebra, it is complete and atomistic ; hence,  $a$  can be written as the join,  $\bigvee_B A$ , of a set  $A$  of atoms in  $B$ . Equivalently,  $a = \bigvee_B \{\bigoplus F \mid F \subseteq A, F \text{ finite}\}$ . By lemma 3.6,  $L$  is orthocomplete, hence,  $\bigoplus A = \bigvee_L \{\bigoplus F \mid F \subseteq A, F \text{ finite}\}$  also exists, and is the limit of the partial sums  $\bigoplus F$ ,  $F \subseteq A$  finite. Since each partial sum lies in  $B$ , and  $B$  is closed,  $\bigoplus A \in B$ . It follows that  $\bigoplus A = a$ . It remains to show that every atom of  $B$  is an atom of  $L$ . Suppose that  $b$  is an atom of  $B$  and that  $x \in L$  with  $0 < x \leq b$ . Since  $B$  is Boolean, every  $y \in B$  satisfies either  $b \leq y$  or  $y \leq b'$ ; thus, either  $x \leq y$  or  $y \leq x$ . In particular,  $x$  is compatible with every element of  $B$ . Since a block in a regular orthoalgebra is a maximal pairwise compatible set,  $x \in B$ , whence,  $x = b$ .  $\square$

**3.3. TOAs with Isolated Zero.** In [3], it is established that any TOML with an isolated point is discrete. In particular, a *compact* TOML with an isolated point is finite. This does not hold for lattice-ordered TOAs generally. Indeed, if  $\mathbf{H}$  is a finite-dimensional Hilbert space, then  $L(\mathbf{H})$  is a compact lattice-ordered TOA in which 0 is isolated. On the other hand, as we now show, a compact TOA with isolated zero does have quite special properties. We begin with an elementary but important observation. Call an open set in a TOA space *totally non-orthogonal* if it contains no two orthogonal elements.

**Proposition 3.13.** *Every non-zero element of a TOA has a totally non-orthogonal open neighborhood.*

*Proof.* Let  $L$  be a TOA. If  $a \neq 0$ , then  $(a, a) \notin \perp$ . Since the latter is closed in  $L^2$ , we can find open sets  $U$  and  $V$  with  $(a, a) \in U \times V$  and  $(U \times V) \cap \perp = \emptyset$ . The set  $U \cap V$  is a totally non-orthogonal open neighborhood of  $a$ .  $\square$

**Proposition 3.14.** *Let  $L$  be a compact TOA with  $0$  isolated. Then*

- (a)  *$L$  is atomistic and of finite height;*
- (b) *The set of atoms of  $L$  is open.*

*Proof.* (a) We first show that there is a finite upper bound on the size of a pairwise orthogonal set. Since  $0$  is isolated in  $L$ ,  $L \setminus \{0\}$  is compact. By Lemma 3.13, we can cover  $L \setminus \{0\}$  by finitely many totally non-orthogonal open sets  $U_1, \dots, U_n$ . A pairwise-orthogonal subset of  $L \setminus \{0\}$  can meet each  $U_i$  at most once, and so, can have at most  $n$  elements. Now given a finite chain  $x_1 < x_2 < \dots < x_m$  in  $L$ , we can construct a pairwise orthogonal set  $y_1, \dots, y_{m-1}$  defined by  $y_1 = x_1$  and  $y_k = x_k \ominus y_{k-1}$  for  $k = 2, \dots, m-1$ . Hence,  $m-1 \leq n$ , so  $m \leq n+1$ . This shows that  $L$  has finite height, from which it follows that  $L$  is atomistic.

(b) Note that if  $A$  and  $B$  are any closed subsets of  $L$ , then  $(A \times B) \cap \perp$  is a closed, hence compact, subset of  $\perp$ . Since  $\oplus$  is continuous on  $\perp$ , the set

$$A \oplus B := \{a \oplus b \mid a \in A, b \in B \text{ and } a \perp b\} = \oplus((A \times B) \cap \perp)$$

is compact, hence closed. The set of non-atoms is precisely  $(L \setminus \{0\}) \oplus (L \setminus \{0\})$ . Since  $0$  is isolated,  $(L \setminus \{0\})$  is closed. Thus, the set of non-atoms is closed.  $\square$

*Remark 3.15.* Notice that both the statements and the proofs of Lemma 3.13 and part (a) of Proposition 3.14 apply verbatim to any topological orthoposet, i.e., any ordered space having a closed order and equipped with a continuous orthocomplementation.

If  $a$  belongs to the center of a TOA  $L$ , then  $[0, a] \times [0, a'] \subseteq \perp$ . Hence, the natural isomorphism  $\phi : [0, a] \times [0, a'] \rightarrow L$  given by  $(x, y) \mapsto x \oplus y$  is continuous. If  $L$  is compact, then so are  $[0, a]$  and  $[0, a']$ ; hence,  $\phi$  is also a homeomorphism. Since the center of an orthoalgebra is a Boolean sub-orthoalgebra of  $L$ , and since a Boolean algebra of finite height is finite, Proposition 3.14 has the following

**Corollary 3.16.** *Let  $L$  be a compact TOA with  $0$  isolated. Then the center of  $L$  is finite. In particular,  $L$  decomposes, both algebraically and topologically, as the product of finitely many compact irreducible TOAs.*

#### 4. STABLY ORDERED TOPOLOGICAL ORTHOALGEBRAS

In this section we consider a particularly tractable, but still quite broad, class of TOAs.

**Definition 4.1.** We shall call an ordered topological space  $L$  *stably ordered* iff, for every open set  $U \subseteq L$ , the upper-set  $U \uparrow = \{b \in L \mid \exists a \in U, a \leq b\}$  is again open.<sup>4</sup>

*Remark 4.2.* Note that this is equivalent to saying that the second projection mapping  $\pi_2 : \leq \rightarrow L$  is an open mapping, since for open sets  $U, V \subseteq L$ ,

$$\pi_2((U \times V) \cap \leq) = U \uparrow \cap V.$$

Note, too, that if  $L$  carries a continuous orthocomplementation  $'$ , then  $L$  is stably ordered iff  $U \downarrow = \{x \mid \exists y \in U, x \leq y\}$  is open for all open sets  $U \subseteq L$ .

<sup>4</sup>The term used by Priestley [13] is “space of type  $I_i$ .”

**Example 4.3.** The following example (a variant of Example 3.3) shows that a TOA need not be stably ordered. Let  $L = [0, 1/4] \cup [3/4, 1]$  with its usual topology, but without its usual order. For  $x, y \in L$ , set  $x \perp y$  iff  $x + y = 1$  or  $x = 0$  or  $y = 0$ . In any of these cases, define  $x \oplus y = x + y$ . As is easily checked, this is a compact lattice-ordered TOA. However, for the clopen set  $[0, 1/4]$  we have  $[0, 1/4]^\uparrow = [0, 1/4] \cup \{1\}$ , which is certainly not open.

Such examples notwithstanding, most of the orthoalgebras that arise “in nature” do seem to be stably ordered. The following is mentioned (without proof) in [13]:

**Lemma 4.4.** *Any topological  $\wedge$ -semilattice – in particular, any topological lattice – is stably ordered.*

*Proof.* If  $L$  is a topological meet-semilattice and  $U \subseteq L$  is open, then

$$U^\uparrow = \{ x \in L \mid \exists y \in U \ x \wedge y \in U \} = \pi_1(\wedge^{-1}(U))$$

where  $\pi_1 : L \times L \rightarrow L$  is the (open) projection map on the first factor and  $\wedge : L \times L \rightarrow L$  is the (continuous) meet operation.  $\square$

Many orthoalgebras, including projection lattices, can be embedded in ordered abelian groups [1]. Indeed, suppose  $G$  is an ordered abelian group. If  $e > 0$  in  $G$ , let  $[0, e]$  denote the set of all elements  $x \in G$  with  $0 \leq x \leq e$ . We can endow  $[0, e]$  with the following partial-algebraic structure: for  $x, y \in [0, e]$ , set  $x \perp y$  iff  $x + y \leq e$ , in which case let  $x \oplus y = x + y$ . Define  $x' = e - x$ . Then  $([0, e], \oplus, ', 0, e)$  is an effect algebra. By a *faithful sub-effect algebra* of  $[0, e]$ , we mean a subset  $L$  of  $[0, e]$ , containing 0 and  $e$ , closed under both  $\oplus$ , where this is defined, and under  $'$ , and such that, for all  $x, y \in L$ ,  $x \leq y$  in  $G$  iff  $\exists z \in L$  with  $y = x + z$  (so that the order inherited from  $G$  and the order induced by  $\oplus$  coincide.)

By way of example, let  $L = L(\mathbf{H})$ , the projection lattice of a Hilbert space  $\mathbf{H}$ , regarded as an orthoalgebra, and let  $G = \mathcal{B}_{sa}(\mathbf{H})$ , the ring of bounded self-adjoint operators on  $\mathbf{H}$  with its operator-norm topology, ordered in the usual way. Then  $L$  is a faithful sub-effect algebra of  $[0, \mathbf{1}]$ , where  $\mathbf{1}$  is the identity operator on  $\mathbf{H}$ . (This follows from the fact that, for projections  $P, Q \in L(\mathbf{H})$ ,  $P + Q \leq \mathbf{1}$  iff  $P \perp Q$ , and the fact that if  $P \leq Q$  as positive operators, then  $Q - P$  is a projection.)

**Lemma 4.5.** *Let  $L$  be an orthoalgebra, let  $G$  be any ordered topological abelian group with a closed cone (equivalently, a closed order), and suppose that  $L$  can be embedded as a sub-effect algebra of  $[0, e]$ , where  $e > 0$  in  $G$ . Then  $L$ , in the topology inherited from  $G$ , is a stably ordered TOA.*

*Proof.* We may assume that  $L$  is a subspace of  $[0, e]$ . Since  $x \perp y$  in  $L$  iff  $x + y \leq e$ , we have  $\perp = +^{-1}([0, e]) \cap L$ , which is relatively closed in  $L$ . The continuity of  $\oplus$  and  $'$  are automatic. Suppose now that  $U \cap L$  is a relatively open subset of  $L$ . Then, since  $L$  is a faithful sub-effect algebra of  $[0, e]$ , the upper set generated by  $U \cap L$  in  $L$  is  $U^\uparrow \cap L$ , where  $U^\uparrow$  is the upper set of  $U$  in  $[0, e]$ . It suffices to show that this last is open. But  $U^\uparrow = \bigcup_{y \geq 0} U + y$ , which is certainly open.  $\square$

In particular, it follows that the projection lattice  $L(\mathbf{H})$  of a Hilbert space  $\mathbf{H}$  is stably ordered in its norm topology.

**Example 4.6.** A *state* on an orthoalgebra  $(L, \oplus)$  is a mapping  $f : L \rightarrow [0, 1]$  such that  $f(\mathbf{1}) = 1$  and, for all  $a, b \in L$ ,  $f(a \oplus b) = f(a) \oplus f(b)$  whenever  $a \oplus b$  exists. A set  $\Delta$  of states on  $L$  is said to be *order-determining* iff  $f(p) \leq f(q)$  for all  $f \in \Delta$

implies  $p \leq q$  in  $L$ . In this case the mapping  $L \rightarrow \mathbb{R}^\Delta$  given by  $p \mapsto \hat{p}$ , where  $\hat{p}(f) = f(p)$ , embeds  $L$  as a faithful sub-effect algebra of the ordered abelian group  $\mathbb{R}^\Delta$  (with pointwise addition, pointwise order). Since the positive cone of  $\mathbb{R}^\Delta$  is closed in the product topology, Lemma 4.5 tells us that  $L$  is stably ordered in the topology inherited from pointwise convergence on  $\Delta$ .

As a special case, the projection lattice  $L = L(\mathbf{H})$  has an order-determining set of states of the form  $f(P) = \langle Px, x \rangle$ , where  $x$  is a unit vector in  $\mathbf{H}$ . If  $\mathbf{H}$  is complex, the polarization identity tells us that the weakest topology on  $L(\mathbf{H})$  making every such state continuous is exactly the restriction to  $L(\mathbf{H})$  of the weak operator topology. Recall that this coincides with the strong operator topology on  $L(\mathbf{H})$ , and makes  $L(\mathbf{H})$  a TOA. Thus, for a complex Hilbert space,  $L(\mathbf{H})$  is stably-ordered in its strong operator topology.

If  $U$  and  $V$  are subsets of an orthoalgebra  $L$ , let us write  $U \oplus V$  for  $\oplus((U \times V) \cap \perp)$ , i.e., for the set of all (existing) orthogonal sums  $a \oplus b$  with  $a \in U$  and  $b \in V$ .

**Lemma 4.7.** *A TOA  $L$  is stably ordered if, and only if, for every pair of open sets  $U, V \subseteq L$ , the set  $U \oplus V$  is also open.*

*Proof.* Let  $U$  and  $V$  be any two open sets in  $L$ . Then

$$\begin{aligned} U \oplus V &= \{c \in L \mid c = a \oplus b, a \in U, b \in V\} \\ &= \{c \in L \mid \exists a \in U \ a \leq c \text{ and } c \ominus a \in V\} \\ &= \pi_2(\ominus^{-1}(V) \cap (L \times U \uparrow)). \end{aligned}$$

Now, since  $L$  is stably ordered,  $U \uparrow$  is open, and hence,  $\ominus^{-1}(V) \cap (L \times U \uparrow)$  is relatively open in  $\leq$ . But as observed above, for  $L$  stably ordered,  $\pi_2 : \leq \rightarrow L$  is an open mapping, so  $U \oplus V$  is open. For the converse, just note that  $U \uparrow = U \oplus L$ .  $\square$

Proposition 3.12 tells us that a compact TOA  $L$  with  $0$  isolated is atomistic and of finite height. It follows easily that every element of  $L$  can be expressed as a finite orthogonal sum of atoms. Let the *dimension*,  $\dim(a)$ , of an element  $a \in L$  be the minimum number  $n$  of atoms  $x_1, \dots, x_n$  such that  $a = x_1 \oplus \dots \oplus x_n$ . Note that  $a \in L$  is an atom iff  $\dim(a) = 1$ .

**Theorem 4.8.** *Let  $L$  be a compact, stably-ordered TOA in which  $0$  is an isolated point. Then*

- (a) *The set of elements of  $L$  of a given dimension is clopen.*
- (b) *The topology on  $L$  is completely determined by that on the set of atoms.*

*Proof.* We begin by noting that if  $A$  and  $B$  are clopen subsets of  $L$ , then  $A \oplus B$  is again clopen (open, by Lemma 4.7, and closed, because the image of the compact set  $(A \times B) \cap \perp$  under the continuous map  $\oplus$ ). Now, since  $0$  is isolated,  $L \setminus \{0\}$  is clopen. Since the set of non-atoms in  $L$  is exactly  $(L \setminus \{0\}) \oplus (L \setminus \{0\})$ , it follows that the set of atoms is clopen. Define a sequence of sets  $L_k$ ,  $k \in \mathbb{N}$ , by setting by  $L_0 = \{0\}$ ,  $L_1$  = the set of atoms of  $L$ , and  $L_{k+1} := L_k \oplus L_1$ . These sets are clopen, as are all Boolean combinations of them. Thus,  $\{a \in L \mid \dim(a) = k\} = L_k \setminus (\bigcup_{i=0}^{k-1} L_i)$  is clopen for every  $k = 0, \dots, \dim(L)$ . This proves (a). For (b), it now suffices to show that the topology on each  $L_k$  is determined by that on  $L_1$ . Since  $L_1$  and  $L_2$  are clopen, Lemma 4.7 tells us that the mapping  $\oplus : ((L_k \times L_1) \cap \perp) \rightarrow L_{k+1}$  is an open surjection, and hence, a quotient mapping. Thus, the topology on  $L_{k+1}$  is entirely determined by that on  $L_k$  and that on  $L_1$ .  $\square$

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