Topological Test Spaces[1]

Alexander Wilce
Department of Mathematical Sciences, Susquehanna University
Selinsgrove, Pa 17870 email: wilce@susqu.edu

Abstract

Test spaces (or manuals) provide a simple, elegant and very general mathematical framework for the study of probability theory – classical, quantum and otherwise. In many particular cases of interest, test spaces carry significant topological structure. This paper inaugurates a general study of topological test spaces. Among other results, we show that any topological test space with a compact space of outcomes is of finite rank. We also generalize recent work of Clifton and Kent by showing that, under very weak assumptions, any second countable topological test space contains a dense semi-classical test space.

0. Introduction

A *test space* in the sense of Foulis and Randall [3, 4, 5], is a pair $(X, \mathfrak{A})$ where X is a non-empty set and $\mathfrak{A}$ is a covering of X by non-empty subsets. [2] The intended interpretation is that each set $E \in \mathfrak{A}$ represents an exhaustive set of mutually exclusive possible *outcomes*, as of some experiment, decision, physical process, or *test*. A *state* or *probability weight* on $(X, \mathfrak{A})$ is a mapping $\omega : X \to [0, 1]$ summing to 1 over each test.

Obviously, this framework subsumes discrete classical probability theory, which may be construed as the theory of test spaces $(E, \{E\})$ having only a single test. It also accommodates quantum probability theory, as follows. Let $\mathbf{H}$ be a Hilbert space, let $S = S(\mathbf{H})$ be the unit sphere of $\mathbf{H}$, and let $\mathfrak{F} = \mathfrak{F}(\mathbf{H})$ denote the collection of all *frames*, i.e., maximal pairwise orthogonal subsets of S. The test space $(S, \mathfrak{F})$ is a model for the set of maximally informative, discrete quantum-mechanical experiments. As long as $\dim(\mathbf{H}) > 2$, Gleason's theorem [6] tells us that every state ω on $(S, \mathfrak{F})$ arises from a density operator W on $\mathbf{H}$ via the formula: $\omega(x) = \langle Wx, x\rangle$ for all $x \in S$.

In this last example, the test space has a natural topological structure: S is a metric space, and $\mathfrak{F}$ can be topologized as well (in several ways). The purpose of this paper is to provide a framework for the study of topological test spaces generally. Section 1 develops basic properties of the Vietoris topology, which we use heavily in the sequel. Section 2 considers topological test spaces in general, and locally finite topological test spaces in particular. Section 3 addresses the problem of topologizing the logic of an algebraic topological test

[1] I wish to dedicate this paper to the memory of Frank J. Hague III

[2] It is also usual to assume that $\mathfrak{A}$ is *irredundant*, i.e., that no set in $\mathfrak{A}$ properly contain another. For convenience, we relax this assumption.

space. In section 4, we generalize recent results of Clifton and Kent [2] by showing that any second-countable topological test space satisfying a rather natural condition contains a dense semi-classical subspace. The balance of this section collects some essential background information concerning test spaces (cf [10] for a detailed survey). Readers familiar with this material can proceed directly to section 1.

0.1 Events Let $(X, \mathfrak{A})$ be a test space. Two outcomes $x, y \in X$ are said to be *orthogonal*, or *mutually exclusive*, if they are distinct and belong to a common test. In this case, we write $x \perp y$. More generally, a set $A \subseteq X$ is called an *event* for X if there exists a test $E \supseteq A$. The set of events is denoted by $\mathcal{E}(X, \mathfrak{A})$.

There is a natural orthogonality relation on $\mathcal{E}(X, \mathfrak{A})$ extending that on X, namely, $A \perp B$ iff $A \cap B = \emptyset$ and $A \cup B \in \mathcal{E}(X, \mathfrak{A})$. Every state ω on $(X, \mathfrak{A})$ extends to a mapping $\omega : \mathcal{E}(X, \mathfrak{A}) \to [0, 1]$ given by $\omega(A) = \sum_{x \in A} \omega(x)$. If $A \perp B$, then $\omega(A \cup B) = \omega(A) + \omega(B)$ for every probability weight ω.

We say that two events A and C of $(X, \mathfrak{A})$ are *complementary*, and write $A\mathsf{oc}C$, if they partition a test. Two events A and B are said to be *perspective* if they are complementary to a common third event C: in this case, we write $A \sim B$. Note that if A and B are perspective, then for every state ω on $(X, \mathfrak{A})$, $\omega(A) = 1 - \omega(C) = \omega(B)$.

For $(S, \mathfrak{F})$, the test space of frames of a Hilbert space $\mathbf{H}$ discussed above, events are simply orthonormal set of vectors in $\mathbf{H}$, and two events are perspective iff they have the same closed span.

0.2 Algebraic Test Spaces We say that X is *algebraic* iff for all events $A, B, C \in \mathcal{E}(X, \mathfrak{A})$,

$$A \sim B \text{ and } B\mathsf{oc}C \;\Rightarrow\; A\mathsf{oc}C.$$

In this case, $\sim$ is an equivalence relation on $\mathcal{E}(X)$. More than this, if $A \perp B$ and $B \sim C$, then $A \perp C$ as well, and $A \cup B \sim A \cup C$. Let $\Pi(X, \mathfrak{A}) = \mathcal{E}(X, \mathfrak{A}) / \sim$, and write $p(A)$ for the $\sim$-equivalence class of an event $A \in \mathcal{E}(X)$. Then Π carries a well-defined orthogonality relation, namely $p(A) \perp p(B) \Leftrightarrow A \perp B$, and also a partial binary operation $p(A) \oplus p(B) = p(A \cup B)$, defined for orthogonal pairs. We may also define $0 := p(\emptyset)$, $1 := p(E)$, $E \in \mathfrak{A}$, and $p(A)' = p(C)$ where C is any event complementary to A. The structure $(\Pi, \oplus, ', 0, 1)$ satisfies the conditions:

(1) $p \oplus q = q \oplus p$ and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$[3];
(2) $p \oplus p$ is defined only if $p = 0$;
(3) $p \oplus 0 = 0 \oplus p = p$;
(4) For every $p \in \Pi$, there exists a unique element — namely, p' — satisfying $p \oplus p' = 1$.

[3] With one side defined iff the other is.

0.3 Orthoalgebras Abstractly, a structure satisfying (1) through (4) above is called an *orthoalgebra*. It can be shown that every orthoalgebra arises canonically (though not uniquely) as $\Pi(X, \mathfrak{A})$ for an algebraic test space $(X, \mathfrak{A})$. Indeed, if L is an orthoalgebra, let $X_L = L \setminus \{0\}$ and let $\mathfrak{A}_L$ denote the set of finite subsets $E = \{e_1, ..., e_n\}$ of $L \setminus 0$ for which $e_1 \oplus \cdots \oplus e_n$ exists and equals 1. Then $(X_L, \mathfrak{A}_L)$ is an algebraic test space with logic canonically isomorphic to L.

Any orthoalgebra L carries a natural partial order, defined by setting $p \leq q$ iff there exists some $r \in L$ with $p \perp r$ and $p \oplus r = q$. With respect to this ordering, the mapping $p \mapsto p'$ is an orthocomplementation.

0.4 Proposition [3]: *If L is an orthoalgebra, the following are equivalent:*

- *(a) L is* orthocoherent*, i.e., for all pairwise orthogonal elements $p, q, r \in L$, $p \oplus q \oplus r$ exists.*
- *(b) $p \oplus q = p \vee q$ for all $p \perp q$ in L*
- *(c) $(L, \leq, ')$ is an orthomodular poset*

Note also that if $(L, \leq, ')$ is any orthoposet, the partial binary operation of orthogonal join — that is, $p \oplus q = p \vee q$ for $p \leq q'$ – is associative iff L is orthomodular, in which case, $(L, \oplus)$ is an orthoalgebra, the natural order on which coincides with the given order on L [10]. Thus, orthomodular posets and orthomodular lattices can be regarded as essentially the same things as orthocoherent orthoalgebras and lattice-ordered orthoalgebras, respectively.

1. Background on the Vietoris Topology

General references for this section are [7] and [8]. If X is any topological space, let 2^X denote the set of all closed subsets of X. If $A \subseteq X$, let

$$[A] := \{F \in 2^X | F \cap A \neq \emptyset\}.$$

Clearly, $[A \cap B] \subseteq [A] \cap [B]$ and $\bigcup_i [A_i] = [\bigcup_i A_i]$. The *Vietoris topology* on 2^X is the coarsest topology in which $[U]$ is open if $U \subseteq X$ is open and $[F]$ is closed if $F \subseteq X$ is closed. [4] Thus, if U is open, so is $(U) := [U^c]^c = \{F \in 2^X | F \subseteq U\}$. Let $\mathcal{B}$ be any basis for the topology on X: then the collection of sets of the form

$$\langle U_1, ..., U_n \rangle \; := \; [U_1] \cap \cdots \cap [U_n] \cap \left(\bigcup_{i=1}^{n} U_i \right)$$

with $U_1, ..., U_n$ in $\mathcal{B}$, is a basis for the Vietoris topology on 2^X. Note that $\langle U_1, ..., U_n \rangle$ consists of all closed sets contained in $\bigcup_{i=1}^n U_i$ and meeting each set U_i at least once.

[4] In particular, $\emptyset$ is an isolated point of 2^X. Many authors omit $\emptyset$ from 2^X.

If X is a compact metric space, then the Vietoris topology on 2^X is just that induced by the Hausdorff metric. Two classical results concerning the Vietoris topology are *Vietoris' Theorem*: 2^X is compact iff X is compact, and *Michael's Theorem*: a (Vietoris) compact union of compact sets is compact.[5]

Note that the operation $\cup : 2^X \times 2^X \to 2^X$ is Vietoris continuous, since

$$\cup^{-1}([U]) = \{(A,B)|A \cup B \in [U]\} = ([U] \times 2^X) \cup (2^X \times [U]),$$

which is open if U is open and closed if U is closed. In particular, for any fixed closed set A, the mapping $f_A : 2^X \to 2^X$ given by $f_A(B) = A \cup B$ is continuous. Notice also that the mapping $\pi : 2^X \times 2^X \to 2^{X \times X}$ given by $\pi(A,B) = A \times B$ is continuous, since $\pi^{-1}([U \times V]) = [U] \times [V]$ and $\pi^{-1}((U \times V)) = (U) \times (V)$.

Henceforth, we regard any collection $\mathfrak{A}$ of closed subsets of a topological space X as a subspace of 2^X. In the special case in which $\mathfrak{A}$ is a collection of finite sets of uniformly bounded cardinality, say $|E| < n$ for every $E \in \mathfrak{A}$, there is a more direct approach to topologizing $\mathfrak{A}$ that bears discussion. Let $\mathfrak{A}^o \subseteq X^n$ denote the space of *ordered* versions $(x_1, ..., x_n)$ of sets $\{x_1, ..., x_n\} \in \mathfrak{A}$, with the relative product topology. We can give $\mathfrak{A}$ the quotient topology induced by the natural surjection $\pi : \mathfrak{A}^o \to \mathfrak{A}$ that "forgets" the order. The following is doubtless well-known, but I include the short proof for completeness.

1.1 Proposition: *Let X be Hausdorff and $\mathfrak{A}$, a collection of non-empty finite subsets of X of cardinality $\leq n$ (with the Vietoris topology). Then the canonical surjection $\pi : \mathfrak{A}^o \to \mathfrak{A}$ is an open continuous map. Hence, the Vietoris topology on $\mathfrak{A}$ coincides with the quotient topology induced by π.*

Proof: Let $U_1, ..., U_n$ be open subsets of X. Then $\pi((U_1 \times \cdots \times U_n) \cap \mathfrak{A}^o) = \langle U_1, ..., U_n \rangle \cap \mathfrak{A}$, so π is open. Also

$$\pi^{-1}(\langle U_1, ..., U_n \rangle \cap \mathfrak{A}) \; = \; \bigcup_{\sigma} (U_{\sigma(1)} \times \cdots \times U_{\sigma(n)}) \cap \mathfrak{A}^o,$$

where σ runs over all permutations of $\{1, 2, ..., n\}$, so π is continuous. It follows immediately that the quotient and Vietoris topologies on $\mathfrak{A}$ coincide. □

2. Topological Test Spaces

We come now to the subject of this paper.

2.1 Definition: A *topological test space* is a test space $(X, \mathfrak{A})$ where X is a Hausdorff space and the relation $\perp$ is closed in $X \times X$.

[5]More precisely, if $\mathcal{C}$ is a compact subset of 2^X with each $C \in \mathcal{C}$ compact, then $\bigcup_{C \in \mathcal{C}} C$ is again compact.

2.2 Examples
(a) Let $\mathbf{H}$ be a Hilbert space. Let S be the unit sphere of $\mathbf{H}$, in any topology making the inner product continuous. Then the test space $(S, \mathfrak{F})$ defined above is a topological test space, since the orthogonality relation is closed in S^2.
(b) More generally, suppose that X is Hausdorff and that $(X, \mathfrak{A})$ is locally finite and supports a set Γ of *continuous* probability weights that are $\perp$-separating in the sense that $p \not\perp q$ iff $\exists \omega \in \Gamma$ with $\omega(p) + \omega(q) > 1$. Then $\perp$ is closed in X^2, so again $(X, \mathfrak{A})$ is a topological test space.
(c) Let L be any topological orthomodular lattice [1]. The mapping $\phi : L^2 \to L^2$ given by $\phi(p, q) = (p, p \wedge q')$ is continuous, and $\perp = \phi^{-1}(\Delta)$ where Δ is the diagonal of L^2. Since L is Hausdorff, Δ is closed, whence, so is $\perp$. Hence, the test space $(L \setminus \{0\}, \mathfrak{A}_L)$ (as described in 0.3 above) is topological.

The following Lemma collects some basic facts about topological test spaces that will be used freely in the sequel.

2.3 Lemma: *Let $(X, \mathfrak{A})$ be a topological test space. Then*
- *(a) Each point $x \in X$ has an open neighborhood containing no two orthogonal outcomes. (We shall call such a neighborhood* totally non-orthogonal.*)*
- *(b) For every set $A \subseteq X$, $A^\perp$ is closed.*
- *(c) Each pairwise orthogonal subset of X is discrete*
- *(d) Each pairwise orthogonal subset of X is closed.*

Proof: (a) Let $x \in X$. Since $(x, x) \notin \perp$ and $\perp$ is closed, we can find open sets V and W about x with $(V \times W) \cap \perp = \emptyset$. Taking $U = V \cap W$ gives the advertised result.

(b) Let $y \in X \setminus x^\perp$. Then $(x, y) \notin \perp$. Since the latter is closed, there exist open sets $U, V \subseteq X$ with $(x, y) \in U \times V$ and $(U \times V) \cap \perp = \emptyset$. Thus, no element of V lies orthogonal to any element of U; in particular, we have $y \in V \subseteq X \setminus x^\perp$. Thus, $X \setminus x^\perp$ is open, i.e., $x^\perp$ is closed. It now follows that for any set $A \subseteq X$, the set $A^\perp = \bigcap_{x \in A} x^\perp$ is closed.

(c) Let D be pairwise orthogonal. Let $x \in D$: by part (b), $X \setminus x^\perp$ is open, whence, $\{x\} = D \cap (X \setminus x^\perp)$ is relatively open in D. Thus, D is discrete.

(d) Now suppose D is pairwise orthogonal, and let $z \in \overline{D}$: if $z \notin D$, then for every open neighborhood U of z, $U \cap D$ is infinite; hence, we can find distinct elements $x, y \in D \cap U$. Since D is pairwise orthogonal, this tells us that $(U \times U) \cap \perp \neq \emptyset$. But then (x, x) is a limit point of $\perp$. Since $\perp$ is closed, $(x, x) \in \perp$, which is a contradiction. Thus, $z \in D$, i.e., D is closed. $\square$

It follows in particular that every test $E \in \mathfrak{A}$ and every event $A \in \mathcal{E}(X, \mathfrak{A})$ is a closed, discrete subset of X. Hence, we may construe $\mathfrak{A}$ and $\mathcal{E}(X, \mathfrak{A})$ of as subspaces of 2^X in the Vietoris topology.

A test space $(X, \mathfrak{A})$ is *locally finite* iff each test $E \in \mathfrak{A}$ is a finite set. We shall say that $(X, \mathfrak{A})$ is of *rank n* if n is the maximum cardinality of a test in $\mathfrak{A}$. If all tests have cardinality *equal* to n, then $(X, \mathfrak{A})$ is *n-uniform.*

2.4 Theorem: *Let $(X, \mathfrak{A})$ be a topological test space with X compact. Then all pairwise orthogonal subsets of X are finite, and of uniformly bounded size. In particular, $\mathfrak{A}$ is of finite rank.*

Proof: By Part (a) of Lemma 2.3, every point $x \in X$ is contained in some totally non- orthogonal open set. Since X is compact, a finite number of these, say $U_1, ..., U_n$, cover X. A pairwise orthogonal set $D \subseteq X$ can meet each U_i at most once; hence, $|D| \leq n$. $\Box$.

For locally finite topological test spaces, the Vietoris topology on the space of events has a particularly nice description. Suppose A is a finite event: By Part (a) of Lemma 2.3, we can find for each $x \in A$ a totally non-orthogonal open neighborhood U_x. Since X is Hausdorff and A is finite, we can arrange for these to be disjoint from one another. Consider now the Vietoris-open neighborhood $\mathcal{V} = \langle U_x, x \in A\rangle \cap \mathcal{E}$ of A in $\mathcal{E}$: an event B belonging to $\mathcal{V}$ is contained in $\bigcup_{x \in A} U_x$ and meets each U_x in at least one point; however, being pairwise orthogonal, B can meet each U_x *at most* once. Thus, B selects *exactly one* point from each of the disjoint sets U_x (and hence, in particular, $|B| = |A|$). Note that, since the totally non-orthogonal sets form a basis for the topology on X, open sets of the form just described form a basis for the Vietoris topology on $\mathcal{E}$.

As an immediate consequence of these remarks, we have the following:

2.5 Proposition: *Let $(X, \mathfrak{A})$ be locally finite. Then the set $\mathcal{E}_n$ of all events of a given cardinality n is clopen in $\mathcal{E}(X, \mathfrak{A})$.*

A test space $(X, \mathfrak{A})$ is *UDF* (*unital, dispersion-free*) iff for ever $x \in X$ there exists a $\{0, 1\}$-valued state ω on $(X, \mathfrak{A})$ with $\omega(x) = 1$. Let $U_1, ..., U_n$ be pairwise disjoint totally non-orthogonal open sets, and and let $\mathcal{U} = \langle U_1, ..., U_n\rangle$: then $\mathcal{U}$ can be regarded as a UDF test space (each U_i selecting one outcome from each test in $\mathcal{V}$). The foregoing considerations thus have the further interesting consequence that any locally finite topological test space is *locally UDF*. In particular, for such test spaces, the existence or non-existence of dispersion-free states will depend entirely on the *global* topological structure of the space.

If $(X, \mathfrak{A})$ is a topological test space, let $\overline{\mathfrak{A}}$ denote the (Vietoris) closure of $\mathfrak{A}$ in 2^X. We are going to show that $(X, \overline{\mathfrak{A}})$ is again a topological test space, having in fact the same orthogonality relation as $(X, \mathfrak{A})$. If $(X, \mathfrak{A})$ is of finite rank, moreover, $(X, \overline{\mathfrak{A}})$ has the same states as $(X, \mathfrak{A})$.

2.6 Lemma: *Let $(X, \mathfrak{A})$ be any topological test space, and let $E \in \overline{\mathfrak{A}}$. Then E is pairwise orthogonal (with respect to the orthogonality induced by $\mathfrak{A}$).*

Proof: Let x and y be two distinct points of E. Let U and V be disjoint neighborhoods of x and y respectively, and let $(E_\lambda)_{\lambda \in \Lambda}$ be a net of closed sets in $\mathfrak{A}$ converging to E in the Vietoris topology. Since $E \in [U] \cap [V]$, we can find $\lambda_{U,V} \in \Lambda$ such that $E_\lambda \in [U] \cap [V]$ for all $\lambda \geq \lambda_{U,V}$. In particular, we can find $x_{\lambda_{U,V}} \in E_{\lambda_{U,V}} \cap U$ and $y_{\lambda_{U,V}} \in E_{\lambda_{U,V}} \cap V$. Since U and V are disjoint, $x_{\lambda_{U,V}}$ and $y_{\lambda_{U,V}}$ are distinct, and hence, – since they belong to a common test E_λ – orthogonal. This gives us a net $(x_{\lambda_{U,V}}, y_{\lambda_{U,V}})$ in $X \times X$ converging to (x, y) and with $(x_{\lambda_{U,V}}, y_{\lambda_{U,V}}) \in \perp$. Since $\perp$ is closed, $(x, y) \in \perp$, i.e., $x \perp y$. □

It follows that the orthogonality relation on X induced by $\overline{\mathfrak{A}}$ is the same as that induced by $\mathfrak{A}$. In particular, $(X, \overline{\mathfrak{A}})$ is again a topological test space.

Let $\mathcal{F}_n$ denote the set of finite subsets of X having n or fewer elements.

2.7 Lemma: *Let X be Hausdorff. Then for every n,*

(a) $\mathcal{F}_n$ is closed in 2^X.

(b) If $f : X \to \mathbb{R}$ is continuous, then so is the mapping $\hat{f} : \mathfrak{F}_n \to \mathbb{R}$ given by $\hat{f}(A) := \sum_{x \in A} f(x)$.

Proof: (a) Let F be a closed set (finite or infinite) of cardinality greater than n. Let $x_1, ..., x_{n+1}$ be distinct elements of F, and let $U_1, ..., U_n$ be pairwise disjoint open sets with $x_i \in U_i$ for each $i = 1, ..., n$. Then no closed set in $\mathcal{U} := [U_1] \cap \cdots \cap [U_n]$ has fewer than $n+1$ points – i.e, $\mathcal{U}$ is an open neighborhood of F disjoint from $\mathcal{F}_n$. This shows that $2^X \setminus \mathcal{F}_n$ is open, i.e., $\mathcal{F}_n$ is closed.

(b) By proposition 1.1, $\mathfrak{F}_n$ is the quotient space of X^n induced by the surjection surjection $q : (x_1, ..., x_n) \mapsto \{x_1, ..., x_n\}$. The mapping $\overline{f} : X^n \to \mathbb{R}$ given by $(x_1, ..., x_n) \mapsto \sum_{i=1}^{n} f(x_i)$ is plainly continuous; hence, so is $\hat{f}$. □

2.8 Proposition: *Let $(X, \mathfrak{A})$ be a rank-n (respectively, n-uniform) test space. Then $(X, \overline{\mathfrak{A}})$ is also a rank-n (respectively, n-uniform) test space having the same continuous states as $(X, \mathfrak{A})$.*

Proof: If $\mathfrak{A}$ is rank-n, then $\mathfrak{A} \subseteq \mathfrak{F}_n$. Since the latter is closed, $\overline{\mathfrak{A}} \subseteq \mathfrak{F}_n$ also. Note that if $\mathfrak{A}$ is n-uniform and $E \in \overline{\mathfrak{A}}$, then any net $E_\lambda \to E$ is eventually in bijective correspondence with E, by Proposition 2.5. Hence, $(X, \overline{\mathfrak{A}})$ is also n-uniform. Finally, every continuous state on $(X, \mathfrak{A})$ lifts to a continuous state on $(X, \overline{\mathfrak{A}})$ by Lemma 2.7 (b). □

3. The Logic of a Topological Test Space

In this section, we consider the logic $\Pi = \Pi(X, \mathfrak{A})$ of an algebraic test space $(X, \mathfrak{A})$. We endow this with the quotient topology induced by the canonical surjection $p : \mathcal{E} \to \Pi$ (where $\mathcal{E} = \mathcal{E}(X, \mathfrak{A})$ has, as usual, its Vietoris topology). Our aim is to find conditions on $(X, \mathfrak{A})$ that will guarantee reasonable continuity properties for the orthogonal sum operation and the orthocomplement. In this connection, we advance the following

3.1 Definition: A *topological orthoalgebra* is an orthoalgbra $(L, \perp, \oplus, 0, 1)$ in which L is a topological space, the relation $\perp \subseteq L^2$ is closed, and the mappings $\oplus : \perp \to L$ and $' : L \to L$ are continuous.

A detailed study of topological orthoalgebras must wait for another paper. However, it is worth mentionioning here that while every topological orthomodular lattice is a topological orthoalgebra, there exist simple examples of lattice-ordered topological orthoalgebras in which the meet and join are discontinuous.

3.2 Lemma: *Let $(L, \perp, \oplus, 0, 1)$ be a topological orthoalgebra. Then*
(a) The order relation $\leq$ is closed in L^2
(b) L is a Hausdorff space.

Proof: For (a), note that $a \leq b$ iff $a \perp b'$. Thus, $\leq = f^{-1}(\perp)$ where $f : L \times L \to L \times L$ is the continuous mapping $f(a, b) = (a, b')$. Since $\perp$ is closed, so is $\leq$. The second statement now follows by standard arguments (cf. Nachbin [9]). □

We now return to the question: when is the logic of a topological test space, in the quotient topology, a topological orthoalgebra?

3.3 Lemma: *Suppose $\mathcal{E}$ is closed in 2^X. Then*
(a) The orthogonality relation $\perp_{\mathcal{E}}$ on $\mathcal{E}$ is closed in $\mathcal{E}^2$.
(b) The mapping $\cup : \perp_{\mathcal{E}} \to \mathcal{E}$ is continuous

Proof: The mapping $\mathcal{E}^2 \to 2^{[X]}$ given by $(A, B) \mapsto A \cup B$ is continuous; hence, if $\mathcal{E}$ is closed in $2^{[X]}$, then so is the set $\mathbf{C} := \{(A, B) \in \mathcal{E}^2 | A \cup B \in \mathcal{E}\}$ of *compatible* pairs of events. It will suffice to show that $\emptyset = \{(A, B) \in \mathcal{E} | A \subseteq B^{\perp}\}$ is also closed, since $\perp = \mathbf{C} \cap \emptyset$. But $(A, B) \in \emptyset$ iff $A \times B \subseteq \perp$, i.e., $\emptyset = \pi^{-1}((\perp)) \cap \mathcal{E}$ where $\pi : 2^X \times 2^X \to 2^{X \times X}$ is the product mapping $(A, B) \mapsto A \times B$. As observed in section 1, this mapping is continuous, and since $\perp$ is closed in $2^{X \times X}$, so is $(\perp)$ in $2^{X \times X}$. Statement (b) follows immediately from the Vietoris continuity of $\cup$. □

Remarks: The hypothesis that $\mathcal{E}$ be closed in 2^X is not used in showing that the relation $\emptyset$ is closed. If $(X, \mathfrak{A})$ is *coherent* [10], $\emptyset = \perp$, so in this case, the hypothesis can be avoided altogether. On the other hand, if X is compact and $\mathfrak{A}$ is closed, then $\mathcal{E}$ will also be compact and hence, closed. (To see this, note that if X is compact then by Vietoris' Theorem, 2^X is compact. Hence, so is the closed set $(E) = \{A \in 2^X | A \subseteq E\}$ for each $E \in \mathfrak{A}$. The mapping $2^X \to 2^{2^X}$ given by $E \mapsto (E)$ is easily seen to be continuous; hence, since $\mathfrak{A}$ is a closed, and so compact, subspace of 2^X, $\{(E) | E \in \mathfrak{A}\}$ is a compact subset of 2^{2^X}. By Michael's theorem, then, $\mathcal{E} = \bigcup_{E \in \mathfrak{A}} (E)$ is compact, hence closed, in 2^X.)

In order to apply Lemma 3.3 to show that $\perp \subseteq \Pi^2$ is closed and $\oplus : \perp \to \Pi$ is continuous, we would like to have the canonical surjection $p : \mathcal{E} \to \Pi$ open. The following condition is sufficient to secure this, plus the continuity of the orthocomplementation $' : \Pi \to \Pi$.

3.3 Definition: Call a topological test space $(X, \mathfrak{A})$ is *stably complemented* iff for any open set $\mathcal{U}$ in $\mathcal{E}$, the set $\mathcal{U}^{\mathsf{oc}}$ of events complementary to events in $\mathcal{U}$ is again open.

Remark: If $\mathbf{H}$ is a finite-dimensional Hilbert space, it can be shown that the corresponding test space $(S, \mathfrak{F})$ of frames is stably complemented (cf. [11]).

3.5 Lemma: *Let $(X, \mathfrak{A})$ be a topological test space, and let $p : \mathcal{E} \to \Pi$ be the canonical quotient mapping (with Π having the quotient topology). Then the following are equivalent:*
(a) $(X, \mathfrak{A})$ is stably complemented
(b) The mapping $p : \mathcal{E} \to \Pi$ is open and the mapping $' : \Pi \to \Pi$ is continuous.

Proof: Suppose first that $(X, \mathfrak{A})$ is stably complemented, and let $\mathcal{U}$ be an open set in $\mathcal{E}$. Then

$$\begin{aligned} p^{-1}(p(\mathcal{U})) &= \{A \in \mathcal{E} | \exists B \in \mathcal{U} A \sim B\} \\ &= \{A \in \mathcal{E} | \exists C \in \mathcal{U}^{\mathsf{oc}} A \mathsf{oc} C\} \\ &= (\mathcal{U}^{\mathsf{oc}})^{\mathsf{oc}} \end{aligned}$$

which is open. Thus, $p(\mathcal{U})$ is open. Now note that $' : \Pi \to \Pi$ is continuous iff, for every open set $V \subseteq \Pi$, the set $V' = \{p' | p \in V\}$ is also open. But $p^{-1}(V') = (p^{-1}(V))^{\mathsf{oc}}$: since p is continuous and $(X, \mathfrak{A})$ is stably complemented, this last is open. Hence, V' is open.

For the converse, note first that if $'$ is continuous, it is also open (since $a'' = a$ for all $a \in \Pi$). Now for any open set $\mathcal{U} \subseteq \mathcal{E}$, $\mathcal{U}^{\mathsf{oc}} = p^{-1}(p(\mathcal{U})')$: Since p and $'$ are continuous open mappings, this last is open as well. □

3.6 Proposition: *Let $(X, \mathfrak{A})$ be a stably complemented algebraic test space with $\mathcal{E}$ closed. Then Π is a topological orthoalgebra.*

Proof: Continuity of $'$ has already been established. We show first that $\perp \subseteq \Pi^2$ is closed. If $(a, b) \not\in \perp$, then for all $A \in p^{-1}(a)$ and $B \in p^{-1}(b)$, $(A, B) \not\in \perp_{\mathcal{E}}$. The latter is closed, by Lemma 3.3 (a); hence, we can find Vietoris-open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of A and B, respectively, with $(\mathcal{U} \times \mathcal{V}) \cap \perp_{\mathcal{E}} = \emptyset$. Since p is open, $U := p(\mathcal{U})$ and $V := p(\mathcal{V})$ are open neighborhoods of a and b with $(U \times V) \cap \perp = \emptyset$. To establish the continuity of $\oplus : \perp \to \Pi$, let $a \oplus b = c$ and let $A \in p^{-1}(a), B \in p^{-1}(B)$ and $C \in p^{-1}(c)$ be representative events. Note that $A \perp B$ and $A \cup B = C$. Let W be an open set containing c: then $\mathcal{W} := p^{-1}(W)$ is an open set containing C. By Lemma 3.3 (b), $\cup : \perp_{\mathcal{E}} \to \mathcal{E}$ is continuous;

hence, we can find open sets $\mathcal{U}$ about A and $\mathcal{V}$ about B with $A_1 \cup B_1 \in \mathcal{W}$ for every $(A_1, B_1) \in (\mathcal{U} \times \mathcal{V}) \cap \perp_{\mathcal{E}}$. Now let $U = p(\mathcal{U})$ and $V = p(\mathcal{V})$: these are open neighborhoods of a and b, and for every $a_1 \in U$ and $b_1 \in V$ with $a_1 \perp b_1$, $a_1 \oplus b_1 \in p(p^{-1}(W)) = W$ (recalling here that p is surjective). Thus, $(U \times V) \cap \perp \subseteq \oplus^{-1}(W)$, so $\oplus$ is continuous. □

5. Semi-classical Test Spaces

From a purely combinatorial point of view, the simplest test spaces are those in which distinct tests do not overlap. Such test spaces are said to be *semi-classical*. In such a test space, the relation of perspectivity is the identity relation on events; consequently, the logic of a semi-classical test space $(X, \mathfrak{A})$ is simply the horizontal sum of the boolean algebras 2^E, E ranging over $\mathfrak{A}$. A state on a semi-classical test space $(X, \mathfrak{A})$ is simply an assignment to each $E \in \mathfrak{A}$ of a probability weight on E. (In particular, there is no obstruction to constructing "hidden variables" models for states on such test spaces.)

Recently, R. Clifton and A. Kent [2] have shown that the test space $(S(\mathbf{H}), \mathfrak{F}(\mathbf{H}))$ associated with a finite-dimensional Hilbert space has (in our language) a dense semi-classical subset. To conclude this paper, I'll show that the this result in fact holds for a large and rather natural class of topological test spaces.

4.1 Lemma: *Let X be any Hausdorff (indeed, T_1) space, and let $U \subseteq X$ be a dense open set. Then $(U) = \{F \in 2^X | F \subseteq U\}$ is a dense open set in 2^X.*

Proof: Since sets of the form $\langle U_1, ..., U_n \rangle$, $U_1, ..., U_n$ open in X, form a basis for the Vietoris topology on 2^X, it will suffice to show that $(U) \cap \langle U_1, ..., U_n \rangle \neq 0$ for all choices of non-empty opens $U_1, ..., U_n$. Since U is dense, we can select for each $i = 1, ..., n$ a point $x_i \in U \cap U_i$. The finite set $F := \{x_1, ..., x_n\}$ is closed (since X is T_1), and by construction lies in $(U) \cap \langle U_1, ..., U_n \rangle$. □

4.2 Corollary: *Let $(X, \mathfrak{A})$ be any topological test space with X having no isolated points, and let E be any test in $\mathfrak{A}$. Then open set $(E^c) = [E]^c$ of tests disjoint from E is dense in $\mathfrak{A}$.*

Proof: Since E is a closed set, its complement E^c is an open set; since E is discrete and includes no isolated point, E^c is dense. The result follows from the preceeding lemma. □

4.3 Theorem: *Let $(X, \mathfrak{A})$ be a topological test space with X (and hence, $\mathfrak{A}$) second countable, and without isolated points. Then there exists a countable, pairwise-disjoint sequence $E_n \in \mathfrak{A}$ such that (i) $\{E_n\}$ is dense in $\mathfrak{A}$, and (ii) $\bigcup_n E_n$ is dense in X.*

Proof: Since it is second countable, $\mathfrak{A}$ has a countable basis of open sets $\mathcal{W}_k$, $k \in \mathbb{N}$. Selecting an element $F_k \in \mathcal{W}_k$ for each $k \in \mathbb{N}$, we obtain a countable dense subset of $\mathfrak{A}$. We shall construct a countable dense pairwise-disjoint subsequence $\{E_j\}$ of $\{F_k\}$. Let $E_1 = F_1$. By Corollary 4.2, $[E_1]^c$ is a dense open set; hence, it has a non-empty intersection with $\mathcal{W}_2$. As $\{F_k\}$ is dense, there exists an index $k(2)$ with $E_2 := F_{k(2)} \in \mathcal{W}_2 \cap [E_1]^c$. We now have $E_1 \in W_1$, $E_2 \in \mathcal{W}_2$, and $E_1 \cap E_2 = \emptyset$. Now proceed recursively: Since $[E_1]^c \cap [E_2]^c \cap \cdots \cap [E_j]^c$ is a dense open and $\mathcal{W}_{j+1}$ is a non-empty open, they have a non-empty intersection; hence, we can select $E_{j+1} = F_{k(j+1)}$ belonging to this intersection. This will give us a test belonging to $\mathcal{W}_{j+1}$ but disjoint from each of the pairwise disjoint sets $E_1, ..., E_j$. Thus, we obtain a sequence $E_j := F_{k(j)}$ of pairwise disjoint tests, one of which lies in each non-empty basic open set $\mathcal{W}_j$ – and which are, ergo, dense.

For the second assertion, it now suffices to notice that for each open set $U \subseteq X$, $[U]$ is a non-empty open in $\mathfrak{A}$, and hence contains some E_j. But then $E_j \cap U \neq \emptyset$, whence, $\bigcup_j E_j$ is dense in X. $\square$

References

[1] Choe, T. H., Greechie, R. J., and Chae, Y., *Representations of locally compact orthomodular lattices*, Topology and its Applications **56** (1994) 165- 173

[2] Clifton, R., and Kent, *Simulating quantum mechanics with non-contextual hidden variables*, The Proceedings of the Royal Society of London A 456 (2000) 2101–2114

[3] Foulis, D. J., Greechie, R. J., and Ruttimann, G. T., *Filters and Supports on Orthoalgebras* Int. J. Theor. Phys. **31** (1992) 789-807

[4] Foulis, D. J., Greechie, R. J., and Ruttimann, G. T., *Logico-algebraic structures II: supports on test spaces*, Int. J. Theor. Phys. **32** (1993) 1675-1690

[5] Foulis, D. J., and Randall, C.H., *A mathematical language for quantum physics*, in H. Gruber et al. (eds.), *Les fondements de la méchanique quantique*, AVCP: Lausanne (1983)

[6] Gleason, A., *Measures on the Closed Subspaces of Hilbert Space*, J. Math. Mech 1957

[7] Illanes, A., and Nadler, S. B., **Hyperspaces**, Dekker: New York (1999)

[8] Michael, E., *Topologies on spaces of subsets*, Trans. Am. Math. Soc. **7** (1951) 152-182

[9] Nachbin, L., **Topology and Order**, van Nostrand: Princeton 1965

[10] Wilce, A., *Test Spaces and Orthoalgebras*, in Coecke et al (eds.) **Current Research in Operational Quantum Logic**, Kluwer: Dordrecht (2000)

[11] Wilce, A., *The Topology of Strongly Symmetric Test Spaces*, in preparation.